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THE

ELEMENTS

OF

EUCLID,

VIZ.

THE FIRST SIX BOOKS,

TOGETHER WITH THE

ELEVENTH AND TWELFTH.

In this Edition, the Errors, by which THEON, or others, have long
ago Vitiated theſe Books, are Corrected,

And ſome of EUCLID's Demonſtrations are Reſtored.

BY

ROBERT SIMSON, M. D.

Profeſſor of Mathematics in the Univerſity of Glaſgow.

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M.DCC.LVI.

TO

HIS ROYAL HIGHNESS

THE PRINCE OF WALES,

WHOSE GENEROUS DISPOSITION

TO ENCOURAGE AND PROMOTE THE ARTS AND SCIENCES,

GIVES THE STRONGEST EVIDENCE HE WILL SOON BECOME

THE GREAT PATRON OF USEFUL LEARNING,

THIS EDITION OF THE PRINCIPAL BOOKS

OF

THE ELEMENTS OF EUCLID,

BY HIS GRACIOUS PERMISSION,

WITH A DUE SENSE OF SO GREAT AN HONOR,

IS MOST HUMBLY DEDICATED

BY HIS ROYAL HIGHNESS'S

MOST OBEDIENT,
MOST DUTIFUL, AND
MOST DEVOTED SERVANT,
ROBERT SIMSON.

PREFACE.

THE Opinions of the Moderns concerning the Author of the Elements of Geometry which go under Euclid's Name, are very different and contrary to one another. Peter Ramus aſcribes the Propoſitions, as well as their Demonſtrations, to Theon; others think the Propoſitions to be Euclid's, but that the Demonſtrations are Theon's; and others maintain that all the Propoſitions and their Demonſtrations are Euclid's own. John Buteo and Sir Henry Savile are the Authors of greateſt Note who aſſert this laſt, and the greater part of Geometers have ever ſince been of this Opinion, as they thought it the moſt probable. Sir Henry Savile, after the ſeveral Arguments he brings to prove it, makes this Concluſion (Pag. 13. Praelect.) "That excepting "a very few Interpolations, Explications and Additions, Theon altered "nothing in Euclid." But, by often conſidering and comparing together the Definitions and Demonſtrations as they are in the Greek Editions we now have, I found that Theon, or whoever was the Editor of the preſent Greek Text, by adding ſome things, ſuppreſſing others, and mixing his own with Euclid's Demonſtrations, had changed more things to the worſe than is commonly ſuppoſed, and thoſe not of ſmall moment, eſpecially in the Fifth and Eleventh Books of the Elements, which this Editor has greatly vitiated. for inſtance, by ſubſtituting a ſhorter, but inſufficient Demonſtration of the 18th Prop. of the 5th Book, in place of the legitimate one which Euclid had given; and by taking out of this Book, beſides other things, the good Definition which Eudoxus or Euclid had given of Compound Ratio, and giving an abſurd one in place of it in the 5th Definition of the 6th Book, which neither Euclid, Archimedes, Apollonius, nor any Geometer before Theon's Time, ever made uſe of, and of which there is not to be found the leaſt appearance in any of their Writings.

and as this Definition did much embarraſs Beginners, and is quite uſeleſs, it is now thrown out of the Elements, and another which without doubt Euclid had given, is put in its proper place among the Definitions of the 5th Book, by which the Doctrine of Compound Ratios is rendered plain and eaſy. Beſides, among the Definitions of the 11th Book, there is this, which is the 10th, viz. "Equal and ſimilar ſolid "figures are thoſe which are contained by ſimilar planes of the ſame "number and magnitude." Now this Propoſition is a Theorem, not a Definition, becauſe the equality of figures of any kind muſt be demonſtrated, and not aſſumed. and therefor, tho' this were a true Propoſition, it ought to have been demonſtrated. But indeed this Propoſition, which makes the 10th Definition of the 11th Book, is not true univerſally, except in the caſe in which each of the ſolid angles of the figures is contained by no more than three plane angles; for, in other caſes, two ſolid figures may be contained by ſimilar planes of the ſame number and magnitude, and yet be unequal to one another; as ſhall be made evident in the Notes ſubjoined to theſe Elements. In the like manner, in the Demonſtration of the 26th Prop. of the 11th Book, it is taken for granted, that thoſe ſolid angles are equal to one another which are contained by plane angles of the ſame number and magnitude placed in the ſame order; but neither is this univerſally true, except in the caſe in which the ſolid angles are contained by no more than three plane angles; nor of this caſe is there any Demonſtration in the Elements we now have, tho' it be quite neceſſary there ſhould be one. Now upon the 10th Definition of this Book depend the 25th and 28th Propoſitions of it; and upon the 25th and 26th depend other eight, viz. the 27th, 31ſt, 32d, 33d, 34th, 36th, 37th, and 40th of the ſame Book. and the 12th of the 12th Book depends upon the 8th of the ſame, and this 8th, and the Corollary of

Propoſition 17th, and Prop. 18th of the 12th Book depend upon the 9th Definition of the 11th Book, which is not a right Definition, becauſe there may be ſolids contained by the ſame number of ſimilar plane figures, which are not ſimilar unto one another, in the true ſenſe of ſimilarity received by all Geometers. and all theſe Propoſitions have, for theſe reaſons, been inſufficiently demonſtrated ſince Theon's time hitherto. Beſides, there are ſeveral other things, which have nothing of Euclid's accuracy, and which plainly ſhew that his Elements have been much corrupted by unſkilful Geometers. and tho' theſe are not ſo groſs as the others now mentioned, they ought by no means to remain uncorrected.

Upon theſe Accounts it appeared neceſſary, and I hope will prove acceptable to all Lovers of Accurate Reaſoning and of Mathematical Learning, to remove ſuch blemiſhes, and reſtore the principal Books of the Elements to their original Accuracy, as far as I was able; eſpecially ſince theſe Elements are the Foundation of a Science by which the Inveſtigation and Diſcovery of uſeful Truths, at leaſt in Mathematical Learning, is promoted as far as the limited Powers of the Mind allow; and which likewiſe is of the greateſt Uſe in the Arts both of Peace and War, to many of which Geometry is abſolutely neceſſary. This I have endeavoured to do by taking away the inaccurate and falſe Reaſonings which unſkilful Editors have put into the Place of ſome of the genuine Demonſtrations of Euclid, who has ever been juſtly celebrated as the moſt accurate of Geometers, and by reſtoring to him thoſe Things which Theon or others have ſuppreſſed, and which have theſe many Ages been buried in Oblivion.

ERRATA.

Page 16. line 1. for concide, read, coincide

P. 51. l. 1. from the bottom, for ABC, read, ABD

P. 131. l. 17. for hat, read, that

P. 174. l. 14. for ratione, read, proportione

P. 210. l. 1. from the bottom, for ſo^{c}, read, ſo^{b}

P. 258. l. 4. for AB, read CB

P. 267. l. 18. after GF, add, and join AH.

P. 277. laſt line, for paralleepiped, read, parallelepiped

P. 297. l. 1. after ſide, add, oppoſite to equal angles

P. 333. line 16. for parallellogram, read, parallelogram

P. 337. l. 5. dele ſuperficies

P. 386. in the title of the Note, after H, add K.

P. 408. l. 7. for CG, read, EG

P. 420. l. 1. for theſe, read, thoſe

P. 423. l. 3. for 34. read, 33.

P. 429. l. 5. dele the comma after στερεῶν

THE
ELEMENTS
OF
EUCLID.

BOOK I.

DEFINITIONS.

I.

A Point is that which hath no parts, or which hath no magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A ſtraight line is that which lies evenly between its extreme points.

V.

A ſuperficies is that which hath only length and breadth.

VI.

The extremities of a ſuperficies are lines.

VII.

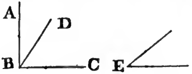
A plain ſuperficies is that in which any two points being taken, the ſtraight line between them lies wholly in that ſuperficies.

VIII.

" A plain angle is the inclination of two lines to one another in a plane, " which meet together, but are not in the ſame direction."

IX.

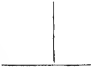
A plain rectilineal angle is the inclination of two ſtraight lines to one another, which meet together, but are not in the ſame ſtraight line.



' N. B. When ſeveral angles are at one point B, any of them is expreſſed by three letters, of which the letter that is at the Vertex of the angle, ' that is at the point in which the ſtraight lines that contain the angle meet ' one another, is put between the other two letters, and one of theſe two ' is ſomewhere upon one of theſe ſtraight lines, and the other upon the ' other line. thus the angle which is contained by the ſtraight lines AB, CB ' is named the angle ABC, or CBA; that which is contained by AB, DB ' is named the angle ABD, or DBA; and that which is contained by DB, ' CB is called the angle DBC, or CBD. but if there be only one angle at a ' point, it may be expreſſed by a letter placed at that point; as the angle ' at E.'

X.

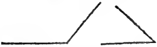
When a ſtraight line ſtanding on another ſtraight line makes the adjacent angles equal to one another, each of theſe angles is called a right angle; and the ſtraight line which ſtands on the other is called a perpendicular to it.



XI.

An obtuſe angle is that which is greater than a right angle.

An



XII.

An acute angle is that which is leſs than a right angle.

XIII.

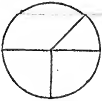
" A term or boundary, is the extremity of any thing."

XIV.

A figure is that which is incloſed by one or more boundaries.

XV.

A circle is a plain figure contained by one line, which is called the circumference, and is ſuch that all ſtraight lines drawn from a certain point within the figure unto the circumference, are equal to one another.



XVI.

And this point is called the centre of the circle.

XVII.

A diameter of a circle is a ſtraight line drawn thro' the centre, and terminated both ways by the circumference.

XVIII.

A ſemicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

" A ſegment of a circle is the figure contained by a ſtraight line and " the circumference it cuts off."

XX.

Rectilineal figures are thoſe which are contained by ſtraight lines.

XXI.

Trilateral figures, or triangles, by three ſtraight lines.

XXII.

Quadrilateral, by four ſtraight lines.

XXIII.

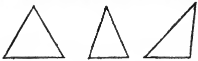
Multilateral figures, or Polygons, by more than four ſtraight lines.

XXIV.

Of three ſided figures, an equilateral triangle is that which has three equal ſides.

XXV.

An iſoſceles triangle, is that which has only two ſides equal.



XXVI.

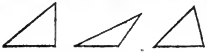
A ſcalene triangle, is that which has three unequal ſides.

XXVII.

A right angled triangle, is that which has a right angle.

XXVIII.

An obtuſe angled triangle, is that which has an obtuſe angle.



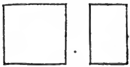
An

XXIX.

An acute angled triangle, is that which has three acute angles.

XXX.

Of four ſided figures, a ſquare is that which has all its ſides equal, and all its angles right angles.

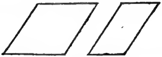


XXXI.

An oblong is that which has all its angles right angles, but has not all its ſides equal.

XXXII.

A rhombus is that which has all its ſides equal, but its angles are not right angles.



XXXIII.

A rhomboeid is that which has its oppoſite ſides equal to one another, but all its ſides are not equal, nor its angles right angles.

XXXIV.

All other four ſided figures beſides theſe, are called Trapeziums.

XXXV.

Parallel, or equidiſtant, ſtraight lines, are ſuch as are in the ſame plane, and which being produced never ſo far both ways, do never meet.

POSTULATES.

I.

LET it be granted that a ſtraight line may be drawn from any one point to any other point.

II.

That a terminated ſtraight line may be produced to any length in a ſtraight line.

III.

And that a circle may be deſcribed from any centre, at any diſtance from that centre.

AXIOMS.

I.

THESE things which are equal to the ſame thing, are equal to one another.

II.

If equal things be added to equals, the wholes ſhall be equal.

III.

If equals be taken from equals, the remainders ſhall be equal.

IV.

If equals be added to unequals, the wholes ſhall be unequal.

V.

If equals be taken from unequals, the remainders ſhall be unequal.

VI.

Things which are double of the ſame, are equal to one another.

VII.

Things which are halves of the ſame, are equal to one another.

Mag-

VIII.

Magnitudes which coincide with one another, that is which exactly fill the ſame ſpace, are equal to one another.

IX.

The whole is greater than its part.

X.

Two ſtraight lines cannot incloſe a ſpace.

XI.

All right angles are equal to one another.

XII.

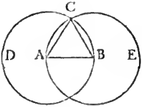
"If a ſtraight line meets two ſtraight lines, ſo as to make the two interior "angles on the ſame ſide of it taken together leſs than two right angles, "theſe ſtraight lines being continually produced ſhall at length meet "upon that ſide on which are the angles which are leſs than two right "angles. See the notes on Prop. 29. of Book I."

PRO-

PROPOSITION I. PROBLEM.

TO deſcribe an equilateral triangle upon a given finite ſtraight line.

Let AB be the given ſtraight line, it is required to deſcribe an equilateral triangle upon AB.



From the centre A, at the diſtance AB
a. 3d Poſtulate. deſcribe [a] the circle BCD. and from the centre B, at the diſtance BA deſcribe the circle ACE; and from the point C in which the circles cut one another draw the ſtraight
b. 2d Poſt. lines [b] CA, CB to the points A, B. ABC ſhall be an equilateral triangle.

Becauſe the point A is the centre of the circle BCD, AC ſhall be e-
c. 15th Definition. qual [c] to AB. and becauſe the point B is the centre of the circle ACE, BC ſhall be equal to BA. but it has been proved that CA is equal to AB; therefor CA, CB are each of them equal to AB. but theſe things
d. 1ſt Axiom. which are equal to the ſame thing are equal to one another [d]; therefor CA is equal to CB. wherefor CA, AB, BC are equal to one another. and the triangle ABC is therefor equilateral, and it is deſcribed upon the given ſtraight line AB. Which was required to be done.

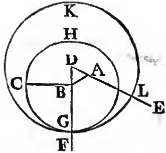
PROP. II. PROB.

TO draw a ſtraight line from a given point that ſhall be equal to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw from the point A a ſtraight line equal to BC.

From the point A to B draw [a] the ſtraight line AB; and upon it deſcribe [b] the equilateral triangle DAB, and produce [c] the ſtraight lines DA, DB to E and F; from the centre B, at the diſtance BC deſcribe [d] the circle CGH, and from the centre D, at the diſtance DG deſcribe the circle GKL. AL ſhall be equal to BC.

a. 1 Poſt.
b. 1. 1.
c. 2. Poſt.
d. 3. Poſt.



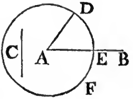
Becauſe the point B is the centre of the circle CGH, BC is equal [e] to BG. and becauſe D is the centre of the circle GKL, DL is equal to DG, and DA, DB parts of them are equal; therefor the remainder AL is equal to the remainder [f] BG. but it has been ſhewn that BC is equal to BG; wherefor AL and BC are each of them equal to BG. and theſe things that are equal to the ſame thing are equal to one another; therefor the ſtraight line AL is equal to BC. Wherefor from the given point A a ſtraight line AL has been drawn equal to the given ſtraight line BC. Which was to be done.

e. 15. Def.
f. 3. Ax.

PROP. III. PROB.

TO cut off from the greater of two given ſtraight lines a part equal to the leſſer.

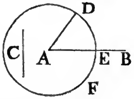
Let AB and C be the two given ſtraight lines, whereof AB is the greater. It is required to cut off from the greater AB a part equal to the leſſer C.



From the point A draw [a] the ſtraight line AD equal to C; and from the centre

a. 2. 1.

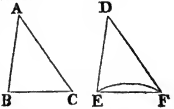
A, and at the diſtance AD deſcribe [b] the circle DEF. and becauſe A
b. 3. Poſt. is the centre of the circle DEF, AE ſhall be equal to AD. but the ſtraight line C is likewiſe equal to AD. whence AE and C are each of them equal to AD. wherefor
c. 1. Ax. the ſtraight line AE is equal [c] to C, and from AB the greater of two ſtraight lines, a part AE ~~has been~~ cut off equal to the leſſer C. Which was to be done.



PROP. IV. THEOREM.

If two triangles have two ſides of the one equal to two ſides of the other, each to each; and have likewiſe the angles contained by theſe ſides equal to one another: they ſhall likewiſe have their baſes, *or third ſides*, equal; and the two triangles ſhall be equal; and their other angles ſhall be equal, each to each, viz. thoſe to which the equal ſides are oppoſite.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; ~~and the angle BAC~~ equal to the angle EDF. the baſe BC ſhall be equal to the baſe EF; and the triangle ABC to the triangle DEF; and the other angles ſhall be equal, each to each, to which the equal ſides are oppoſite, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.



For if the triangle ABC be applied to DEF ſo that the point A may be in D, and the ſtraight line AB upon DE; the point B ſhall coincide with the point E, becauſe AB is equal to DE. and AB coinciding with DE, AC ſhall coincide with DF, becauſe the angle BAC is equal to the angle EDF. wherefor alſo the point C ſhall coincide with the point F, becauſe the ſtraight line AC is equal to DF. but the point B coincides with the point E; wherefor the baſe BC ſhall coincide with the baſe EF. becauſe the point B coinciding with E, and C with F, if the baſe BC does not coincide with the baſe EF, two ſtraight lines could incloſe a ſpace, which is impoſſible[a]. Therefor the baſe BC ſhall a. 10. Ax. coincide with the baſe EF, and be equal to it. Wherefor the whole triangle ABC ſhall coincide with the whole triangle DEF, and be equal to it; and the other angles of the one ſhall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to DFE. Therefor if two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe the angles contained by theſe ſides equal to one another; their baſes ſhall likewiſe be equal, and the triangles ſhall be equal, and their other angles ſhall be equal, each to each, to which the equal ſides are oppoſite. Which was to be demonſtrated.

PROP. V. THEOR.

THE angles at the baſe of an Iſoſceles triangle are equal to one another; and if the equal ſides be produced, the angles upon the other ſide of the baſe ſhall be equal.

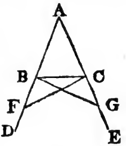
Let ABC be an Iſoſceles triangle, whoſe ſide AB is equal to AC, and let the ſtraight lines AB, AC be produced to D and E. the angle ABC ſhall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from the greater AE cut off AG equal [a] to the leſſer AF, and join FC, GB. a. 3. 1.

Becauſe AF is equal to AG, and AB to AC; the two ſides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefor the baſe FC is equal [b] to the baſe GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one ſhall be equal [b] to the remaining angles of the other, each to each, to which the equal ſides are oppoſite; viz. the angle ACF to the angle ABG, and the angle AFC to the angle AGB. and becauſe the whole AF is equal to the whole AG, of which the parts AB, AC are equal; the remainder BF ſhall be equal [c] to the remainder CG. and FC was proved to be equal to GB; therefor the two ſides BF, FC are equal to the two CG, GB, each to each; and the angle BFC is equal to the angle CGB; and the baſe BC is common to the two triangles BFC, CGB; wherefor theſe triangles are equal [b], and their remaining angles, each to each, to which the equal ſides are oppoſite. therefor the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG. and ſince it has been demonſtrated that the whole angle ABG is equal to the whole ACF, the parts of which, the angles CBG, BCF are alſo equal; the remaining angle ABC ſhall be equal to the remaining angle ACB, which are the angles at the baſe of the triangle ABC. and it has alſo been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other ſide of the baſe. Therefor the angles at the baſe, &c. Q. E. D.

b. 4. 1.

c. 3. Ax.



Corollary. Hence every equilateral triangle is alſo equiangular.

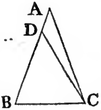
PROP. VI.

PROP. VI. THEOR.

IF two angles of a triangle be equal to one another, likewiſe the ſides which ſubtend, *or are oppoſite to*, the equal angles ſhall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the ſide AB ſhall likewiſe be equal to the ſide AC.

For if AB be not equal to AC, one of them is greater than the other. let AB be the greater, and from it cut [a] off DB equal to the leſſer AC, and join DC. therefor becauſe in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two ſides DB, BC ſhall be equal to the two AC, CB, each to each; and the angle DBC is equal to the angle ACB; therefor the baſe DC is equal to the baſe AB, and the triangle DBC, equal to the triangle [b] ACB, the leſs to the greater; which is abſurd. Therefor AB is not unequal to AC, that is, it is equal to it. Wherefor if two angles, &c. Q. E. D.



Cor. Hence every equiangular triangle is alſo equilateral.

PROP. VII. THEOR.

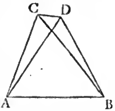
THERE cannot be two triangles upon the ſame baſe, and upon the ſame ſide of it, that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity.

If

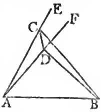
[a] 3. 1.

[b] 4. 1.

If it be poſſible, let there be two triangles ACB, ADB upon the ſame baſe AB, and upon the ſame ſide of it, which have their ſides CA, DA, that are terminated in the extremity A of the baſe, equal to one another, and likewiſe their ſides CB, DB that are terminated in B.



Join CD; then, in the caſe in which the Vertex of each of the triangles is without the other triangle, becauſe AC is equal to AD, a. 5. 1. the angle ACD ſhall be equal [a] to the angle ADC. but the angle ACD is greater than the angle BCD, therefor the angle ADC is greater alſo than BCD; much more then is the angle BDC greater than the angle BCD. again, becauſe CB is equal to DB, the angle BDC ſhall be equal [a] to the angle BCD; but it has been demonſtrated to be greater than it; which is impoſſible.



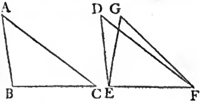
But if one of the Vertices, as D, be within the other triangle ACB; produce AC, AD to E, F. therefor becauſe AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other ſide of the baſe CD ſhall be equal [a] to one another; but the angle ECD is greater than the angle BCD, wherefor the angle FDC is likewiſe greater than BCD; much more then is the angle BDC greater than the angle BCD. again becauſe CB is equal to DB, the angle BDC ſhall be equal [a] to the angle BCD; but BDC has been proved to be greater than the ſame BCD, which is impoſſible. The caſe in which the Vertex of one triangle is upon a ſide of the other, needs no demonſtration.

Therefor there cannot be two triangles upon the ſame baſe, and upon the ſame ſide of it, that have their ſides which are terminated in one of the extremities of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity. Q. E. D.

PROP. VIII. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe their baſes equal; the angle which is contained by the two ſides of the one ſhall be equal to the angle contained by the two ſides equal to them, of the other.

Let ABC, DEF be two triangles having the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; and alſo the baſe BC equal to the baſe EF. The angle BAC ſhall be equal to the angle EDF.



For if the triangle ABC be applied to DEF ſo that the point B be in E, and the ſtraight line BC upon EF; the point C ſhall alſo coincide with the point F, becauſe BC is equal to EF. therefor BC coinciding with EF, BA and AC ſhall coincide with ED and DF. for if the baſe BC coincides with the baſe EF, but the ſides BA, CA do not coincide with the ſides ED, FD, but have a different ſituation, as EG, FG; then upon the ſame baſe EF and upon the ſame ſide of it there can be two triangles which have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe their ſides terminated in the other extremity. but this is impoſſible[a]. therefor if the baſe BC coincides with the baſe EF, the ſides BA, AC cannot but coincide with the ſides ED, DF; wherefor likewiſe the angle

BAC

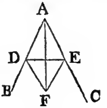
[a] 7. 1.

BAC ſhall concide with the angle EDF, and be equal [b] to it. there-
b. 8. Ax. for if two triangles, &c. Q. E. D.

PROP. IX. PROB.

TO biſſect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to biſſect it.
a. 3. 1. Take any point D in AB, and from AC cut [a] off AE equal to AD;
b. 1. 1. join DE, and upon it deſcribe [b] an equilateral triangle DEF, then join AF. the ſtraight line AF ſhall biſſect the angle BAC.



Becauſe AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two ſides DA, AF are equal to the two ſides EA, AF, each to each; and the baſe DF is equal to
c. 8. 1. the baſe EF; therefor the angle DAF is equal [c] to the angle EAF. wherefor the given rectilineal angle BAC is biſſected by the ſtraight line AF. Which was to be done.

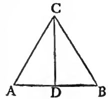
PROP. X. THEOR.

TO biſſect a given finite ſtraight line, that is, to divide it into two equal parts.

Let AB be the given ſtraight line; it is required to divide it into two equal parts.
a. 1. 1. Deſcribe [a] upon it an equilateral triangle ABC, and biſſect [b] the angle
b. 9. 1. ACB by the ſtraight line CD. AB ſhall be cut into two equal parts in the point D.

Becauſe AC is equal to CB, and CD common to the two triangles ACD, BCD; the two ſides AC, CD are equal to BC, CD, each to each; and the angle ACD is equal to the angle BCD; therefor the baſe AD is equal to the baſe [c] DB, and the ſtraight line AB is divided into two equal parts in the point D. Which was to be done.

c. 4. 1.



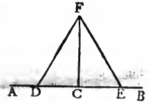
PROP. XI. PROB.

TO draw a ſtraight line at right angles to a given ſtraight line, from a given point in the ſame.

Let AB be a given ſtraight line, and C a point given in it; it is required to draw a ſtraight line from the point C at right angles to AB.

Take any point D in AC, and make [a] CE equal to CD, and upon DE deſcribe [b] the equilateral triangle DFE, and join FC. the ſtraight line FC drawn from the given point C, ſhall be at right angles to the given ſtraight line AB.

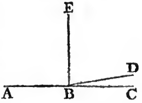
a. 3. 1.
b. 1. 1.



Becauſe DC is equal to CE, and FC common to the two triangles DCF, ECF; the two ſides DC, CF are equal to the two EC, CF, each to each; and the baſe DF is equal to the baſe EF; therefor the angle DCF is equal [c] to the angle ECF; and they are adjacent angles. but when the adjacent angles which one ſtraight line makes with another ſtraight line are equal to one another, each of them is called a right [d] angle; therefor each of the angles DCF, ECF is a right angle. wherefor from the given point C in the given ſtraight line AB, FC has been drawn at right angles to AB. Which was to be done.

c. 8. 1.
d. 10. Def. 1.

Cor. By help of this Problem it may be demonſtrated that two ſtraight lines cannot have a common ſegment.

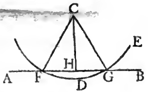


If it be poſſible, let the two ſtraight lines ABC, ABD have the ſegment AB common to both of them. from the point B draw BE at right angles to AB; and becauſe ABC
d. 10. Def. 1. is a ſtraight line, the angle CBE is equal [d] to the angle EBA; in the ſame manner, becauſe ABD is a ſtraight line, the angle DBE is equal to the angle EBA. wherefor the angle DBE is equal to the angle CBE, the leſs to the greater; which is impoſſible. therefor two ſtraight lines cannot have a common ſegment.

PROP. XII. PROB.

To draw a ſtraight line perpendicular to a given ſtraight line of an unlimited length, from a given point without it.

Let AB be the given ſtraight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a ſtraight line perpendicular to AB from the point C.



Take any point D upon the other ſide of AB, and from the centre C, at the diſ-
a. 3. Poſt. tance CD, deſcribe [a] the circle EGF meet-
b. 10. 1. ing AB in F, G; and biſſect [b] FG in H, and join CF, CH, CG. the ſtraight line CH drawn from the given point C, ſhall be perpendicular to the given ſtraight line AB.

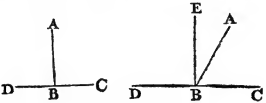
Becauſe FH is equal to HG, and HC common to the two triangles

FHC, GHC, the two ſides FH, HC are equal to the two GH, HC,
each to each; and the baſe CF is equal[c] to the baſe CG; therefor c. 15. Def. 1.
the angle CHF is equal[d] to the angle CHG; and they are adjacent d. 8. 1.
angles. but when a ſtraight line ſtanding on a ſtraight line makes the adjacent angles equal to one another, each of them is a right angle, and the ſtraight line which ſtands upon the other is called a perpendicular to it. therefor from the given point C a perpendicular CH has been drawn to the given ſtraight line AB. Which was to be done.

PROP. XIII. THEOR.

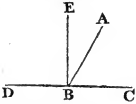
THE angles which one ſtraight line makes with another upon one ſide of it, are either two right angles, or are together equal to two right angles.

Let the ſtraight line AB make with CD, upon one ſide of it, the angles CBA, ABD; theſe ſhall either be two right angles, or ſhall together be equal to two right angles.



For if the angle CBA be equal to ABD, each of them is a right[a] a. Def. 10.
angle. but if not, from the point B draw BE at right angles[b] to CD. b. 11. 1.
therefor the angles CBE, EBD are two right angles[a]. and becauſe CBE
is equal to the two angles CBA, ABE together; add the angle EBD to
each of theſe equals, and the angles CBE, EBD ſhall be equal[c] to c. 2. Ax.
the

the three angles CBA, ABE, EBD. again, becauſe the angle DBA is equal to the two angles DBE, EBA, add to theſe equals the angle ABC; and the angles DBA, ABC ſhall be equal to the three angles DBE, EBA, ABC. but the angles CBE, EBD have been demonſtrated to be equal to the ſame three angles; and thoſe things that are equal to the ſame thing are equal [d] to one another; therefor the angles CBE, EBD are equal to the angles DBA, ABC. but CBE, EBD are two right angles; therefor DBA, ABC are together equal to two right angles. Wherefor when a ſtraight line &c. Q. E. D.

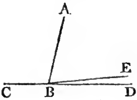


d. 1. Ax.

PROP. XIV. THEOR.

IF at a point in a ſtraight line, two other ſtraight lines, upon the oppoſite ſides of it, make the adjacent angles together equal to two right angles, theſe two ſtraight lines ſhall be in one and the ſame ſtraight line.

At the point B in the ſtraight line AB, let the two ſtraight lines, BC, BD upon ~~the oppoſite ſides of AB,~~ make the adjacent angles ABC, ABD equal together to two right angles. BD ſhall be in the ſame ſtraight line with CB.



For if BD be not in the ſame ſtraight line with CB, let BE be in the ſame ſtraight line with it. therefor becauſe the ſtraight line AB makes angles with the ſtraight line CBE, upon one ſide of it, theſe angles ABC, ABE ſhall together be equal [a] to two right angles. but the angles ABC, ABD are likewiſe together equal to two right angles. therefor the angles CBA, ABE are equal to the angles CBA, ABD. take away the common angle ABC, and the remaining angle ABE ſhall be equal [b] to the remaining angle ABD, the leſſer to the greater, which is impoſſible. therefor BE is not in the ſame ſtraight line with BC. And in like manner, it ſhall be demonſtrated that no other can be but BD, which therefor is in the ſame ſtraight line with CB. Wherefor if at a point &c. Q. E. D.

a. 13. 1.
b. 3. Ax.

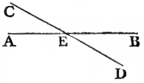
PROP. XV. THEOR.

IF two ſtraight lines cut one another, the vertical, *or oppoſite*, angles ſhall be equal.

Let the two ſtraight lines AB, CD cut one another in the point E. the angle AEC ſhall be equal to the angle DEB, and CEB to AED.

Becauſe the ſtraight line AE makes with CD the angles CEA, AED, theſe angles ſhall together be equal [a] to two right angles. again, becauſe the ſtraight line DE makes with AB the angles AED, DEB; theſe alſo ſhall together be equal [a] to two right angles. and CEA, AED have been demonſtrated to be equal to two right angles; wherefor the angles CEA, AED are equal to the angles AED, DEB. take away the common angle AED, and the remaining angle CEA ſhall be equal [b] to the remaining angle DEB. In the ſame manner it ſhall be demonſtrated that the angles CEB, AED are equal. therefor if two ſtraight lines &c. Q. E. D.

a. 13. 1.
b. 3. Ax.



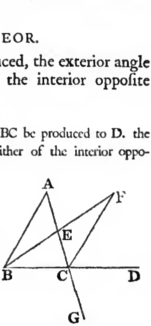
Cor. 1. From this it is manifeſt that if two ſtraight lines cut one another, the angles they make at the point where they cut, ſhall together be equal to four right angles.

Cor. 2. And conſequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP. XVI. THEOR.

IF one ſide of a triangle be produced, the exterior angle ſhall be greater than either of the interior oppoſite angles.

Let ABC be a triangle, and let its ſide BC be produced to D. the exterior angle ACD ſhall be greater than either of the interior oppoſite angles CBA, BAC.



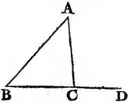
a. 10. 1. Biſſect [a] AC in E, join BE and produce it to F, and make EF equal to BE; join alſo FC, and produce AC to G.

Becauſe AE is equal to EC, and BE to EF; AE, EB are equal to CE, EF, each to each; and the angle AEB is b. 15. 1. equal [b] to the angle CEF, becauſe they are oppoſite vertical angles. therefor the c. 4. 1. baſe AB is equal [c] to the baſe CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal ſides are oppoſite. wherefor the angle BAE is equal to the angle ECF. but the angle ECD is greater than the angle ECF, therefor the angle ACD is greater than BAE. in the ſame manner, if the ſide BC be biſſected, it ſhall be demonſtrated that the angle BCG, that is[d], the angle ACD, is greater than the angle ABC. therefor if one ſide &c. Q. E. D. d. 15. 1.

PROP. XVII. THEOR.

ANY two angles of a triangle are together leſs than two right angles.

Let ABC be any triangle; any two of its angles together ſhall be leſs than two right angles.

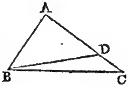


Produce BC to D; and becauſe ACD is the exterior angle of the triangle ABC, ACD ſhall be greater [a] than the interior and oppoſite angle ABC; to each of theſe add the angle ACB, and the angles ACD, ACB ſhall be greater than the angles ABC, ACB. but ACD, ACB are together equal [b] to two right angles; therefor the angles ABC, BCA are leſs than two right angles. in like manner it ſhall be demonſtrated that BAC, ACB, as alſo CAB, ABC are leſs than two right angles. therefor any two angles &c. Q. E. D.

a. 16. 1.
b. 13. 1.

PROP. XVIII. THEOR.

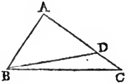
THE greater ſide of every triangle is oppoſite to the greater angle.



Let ABC be a triangle of which the ſide AC is greater than the ſide AB; the angle ABC ſhall likewiſe be greater than the angle BCA.

Becauſe AC is greater than AB, make [a] a. 3. 1.

AD equal to AB, and join BD. and becauſe ADB is the exterior angle
b. 16. 1. of the triangle BDC, ADB ſhall be greater [b]
than the interior and oppoſite angle DCB.
c. 5. 1. but ADB is equal [c] to ABD, becauſe the
ſide AB is equal to the ſide AD; therefor
the angle ABD is likewiſe greater than the
angle ACB; and much more ſhall the angle
ABC be greater than ACB. therefor the
greater ſide &c. Q. E. D.

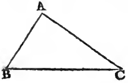


PROP. XIX. THEOR.

THE greater angle of every triangle is ſubtended by the greater ſide, or *has the greater ſide oppoſite to it.*

Let ABC be a triangle of which the angle ABC is greater than the angle BCA. the ſide AC ſhall likewiſe be greater than the ſide AB.

For if it be not greater, AC muſt either be equal to AB, or leſs
than it. but it is not equal, becauſe then the
a. 5. 1. angle ABC would be equal [a] to the angle
ACB; but it is not; therefor AC is not e-
qual to AB. but neither is it leſs than it;
becauſe then the angle ABC would be leſs
b. 18. 1. [b] than the angle ACB; but it is not; there-
for the ſide AC is not leſs than AB. and it has been ſhewn that it
is not equal to AB. therefor AC is greater than AB. wherefor the
greater angle &c. Q. E. D.

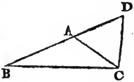


PROP. XX.

PROP. XX. THEOR.

ANY two ſides of a triangle are together greater than the third ſide.

Let ABC be a triangle; any two ſides of it together ſhall be greater than the third ſide, viz. the ſides BA, AC ſhall be greater than the ſide BC; and AB, BC greater than AC; and BC, CA greater than AB.



Produce BA to the point D, and make [a] AD equal to AC, and join DC. a. 3. 1.

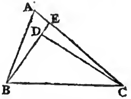
Becauſe DA is equal to AC, the angle ADC ſhall be equal [b] to ACD. b. 5. 1. but the angle BCD is greater than the angle ACD; therefor the angle BCD is greater than the angle ADC. and becauſe the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater [c] ſide is oppo- c. 19. 1. ſite to the greater angle, the ſide DB ſhall be greater than the ſide BC. but DB is equal to BA and AC; therefor the ſides BA, AC are greater than BC. in the ſame manner it ſhall be demonſtrated that the ſides AB, BC are greater than CA; and BC, CA greater than AB. therefor any two ſides &c. Q. E. D.

PROP. XXI. THEOR.

IF from the ends of the ſide of a triangle there be drawn two ſtraight lines to a point within the triangle, theſe ſhall be leſs than the other two ſides of the triangle, but ſhall contain a greater angle.

Let the two ſtraight lines BD, CD be drawn from B, C the ends of the ſide BC of the triangle ABC to the point D within it. BD and DC ſhall be leſſer than the other two ſides BA, AC of the triangle, but ſhall contain an angle BDC greater than the angle BAC.

Produce BD to E; and becauſe two ſides of a triangle are greater than the third ſide, the two ſides BA, AE of the triangle ABE ſhall be greater than BE. to each of theſe add EC, and the ſides BA, AC ſhall be greater than BE, EC. again, becauſe the two ſides CE, ED of the triangle CED are greater than CD, add DB to each of theſe, and the ſides CE, EB ſhall be greater than CD, DB. but it has been ſhewn that BA, AC, are greater than BE, EC; much more then are BA, AC greater than BD, DC.



Again becauſe the exterior angle of a triangle is greater than the interior and oppoſite angle, the exterior angle BDC of the triangle CDE ſhall be greater than CED. by the ſame reaſon, the exterior angle CEB of the triangle ABE is greater than BAC. and it has been demonſtrated that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. therefor if from the ends of &c. Q. E. D.

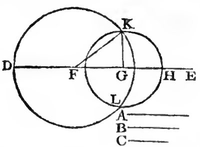
PROP. XXII. PROB.

TO make a triangle whoſe ſides ſhall be equal to three given ſtraight lines; but any two whatever of theſe muſt be greater than the third[1].

a. 20. 1

Let A, B, C be the three given ſtraight lines, of which any two whatever are greater than the third, viz. A and B greater than C; A and C greater than B; and B and C than A. It is required to make a tria triangle whoſe ſides ſhall be equal to A, B, C, each to each.

Take a ſtraight line DE terminated at the point D, but unlimited towards E, and make [a] DF equal to A, FG to B, and GH equal to a. 3. 1. C; and from the centre F, at the diſtance FD deſcribe [b] the circle b. 3. Poſt. DKL. and from the centre G, at the diſtance GH deſcribe [b] another circle HLK, and join KF, KG. the triangle KFG ſhall have its ſides equal to the three ſtraight lines A, B, C.

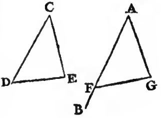


Becauſe the point F is the centre of the circle DKL, FD is equal [c] c. 15. Def. to FK; but FD is equal to the ſtraight line A; therefor FK is equal to A. again, becauſe G is the centre of the circle LKH, GH is equal [c] to GK; but GH is equal to C, therefor alſo GK is equal to C. and FG is equal to B; therefor the three ſtraight lines KF, FG, GK are equal to the three A, B, C. and therefor the triangle KFG has its three ſides KF, FG, GK equal to the three given ſtraight lines A, B, C. Which was to be done.

PROP. XXIII. PROB.

AT a given point in a given ſtraight line to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given ſtraight line AB that ſhall be equal to the given rectilineal angle DCE.



Take in CD, CE, any points D, a. 22. 1. E, and join DE; and make [a] the triangle AFG whoſe ſides ſhall be equal to the three ſtraight lines CD, DE, EC, ſo that CD be equal to AF, CE to AG, and DE to FG. and becauſe DC, CE are equal to FA, AG, each to each, and the baſe DE to the baſe FG; b. 8. 1. the angle DCE ſhall be equal [b] to the angle FAG. therefor at the given point A in the given ſtraight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

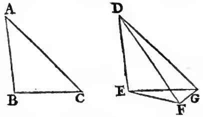
PROP. XXIV. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the angle contained by the two ſides of one of them greater than the angle contained by the two ſides equal to them, of the other; the baſe of that which has the greater angle ſhall be greater than the baſe of the other.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF. the baſe BC ſhall be greater than the baſe EF.

Of the two ſides DE, DF let DE be the ſide which is not greater than the other, and at the point D in the ſtraight line DE make[a] the angle EDG equal to the angle BAC; and make DG equal[b] to AC or DF, and join EG, GF.

a. 23. 1.
b. 3. 1.



Becauſe AB is equal to DE, and AC to DG, the two ſides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefor the baſe BC is equal[c] to the baſe EG. and becauſe DG is equal to DF, the angle DFG is equal[d] to the angle DGF; but the angle DGF is greater than the angle EGF, and therefor the angle DFG is greater than EGF; and much more ſhall the angle EFG be greater than the angle EGF. and becauſe the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater[e] ſide is oppoſite to the greater angle; the ſide EG ſhall be greater than the ſide EF. but EG is equal to BC; and therefor alſo BC ſhall be greater than EF. therefor if two triangles &c. Q. E. D.

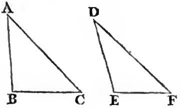
c. 4. 1.
d. 5. 1.
e. 19. 1.

PROP. XXV. THEOR.

IF two triangles have two ſides of one equal to two ſides of the other, each to each, but the baſe of one greater than the baſe of the other; likewiſe the angle contained by the ſides of that which has the greater baſe, ſhall be greater than the angle contained by the ſides equal to them, of the other.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the baſe CB greater than the baſe EF. the angle BAC ſhall likewiſe be greater than the angle EDF.

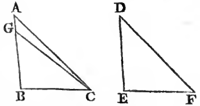
For if it be not greater, it muſt either be equal to it, or leſſer. but the angle BAC is not equal to the angle EDF, becauſe then the baſe a. 4. 1. BC would be equal [a] to EF. but it is not; therefor the angle BAC is not equal to the angle EDF. but neither is it leſſer; becauſe then the baſe BC would be leſſer than the baſe EF; but it is not; therefor the angle BAC is not leſs than the angle EDF. and it was ſhewn that it is not equal to it; therefor the angle BAC is greater than the angle EDF. wherefor if two triangles &c. Q. E. D.



PROP. XXVI. THEOR.

IF two triangles have two angles of one equal to two angles of the other; each to each, and one ſide to one ſide, viz. either the ſides adjacent to the equal angles, or the ſides oppoſite to two of them in the two triangles; then ſhall the other ſides be equal, each to each, and alſo the third angle of the one to the third angle of the other.

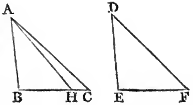
Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; alſo one ſide equal to one ſide; and firſt, let theſe ſides be equal which are adjacent to the angles which are equal in the two triangles, viz. BC to EF. the other ſides ſhall be equal, each to each, viz. AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.



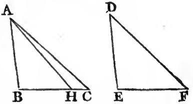
For if AB be not equal to DE, one of them ſhall be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC. therefor becauſe BG is equal to DE, and BC to EF, the two ſides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefor the baſe GC is equal [a] to the baſe DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal ſides are oppoſite; therefor the angle GCB is equal to the angle DFE; but DFE is, by the hypotheſis, equal to the angle BCA; wherefor alſo the angle BCG is equal to the angle BCA, the leſs to the greater, which is impoſſible. therefor AB is not unequal to DE, that is, it is equal to it. and BC is equal to EF; therefor the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF, the baſe therefor AC is equal [a] to the baſe DF, and the third angle BAC to the third angle EDF.

a. 4. 1.

Next, let the ſides which are oppoſite to two of the equal angles in the two triangles be equal to one another, viz. AB to DE; likewiſe in this caſe, the other ſides ſhall be equal, AC to DF, and BC to EF; and alſo the third angle BAC to the third EDF.



For if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH. and becauſe BH is equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefor the baſe AH is equal to the baſe DF, and the triangle ABH to the triangle DEF, and the other angles ſhall be equal, each to each, to which the equal ſides are oppoſite. therefor the angle BHA is equal to the angle EFD. but EFD is equal to the angle BCA; therefor alſo the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and oppoſite angle BCA; which is impoſſible[b]. wherefor BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefor the two AB, BC are equal to the two DE, EF, each to each; and they contain equal angles; wherefor the baſe AC is equal to the baſe DF, and the third angle BAC to the third angle EDF. therefor if two triangles &c. Q. E. D.



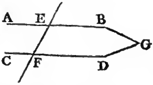
b. 16. 1.

PROP. XXVII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the alternate angles equal to one another, theſe two ſtraight lines ſhall be parallel.

Let the ſtraight line EF which falls upon the two ſtraight lines AB, CD make the alternate angles AEF, EFD equal to one another; AB ſhall be parallel to CD.

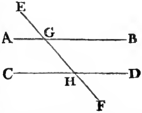
For if it be not parallel, AB and CD being produced ſhall meet either towards BD or towards AC. let them be produced and meet towards BD in the point G; therefor GEF is a triangle, and its exterior angle AEF ſhall be greater [a] than the interior and oppoſite angle EFG; a. 16. 1. but it is alſo equal to it, which is impoſſible. therefor AB and CD being produced do not meet towards BD. in like manner it ſhall be demonſtrated that they do not meet towards AC. but thoſe ſtraight lines which meet neither way tho' produced never ſo far are parallel [b] to one another. b. 35. Def. AB therefor is parallel to CD. wherefor if a ſtraight line &c. Q. E. D.



PROP. XXVIII. THEOR.

IF a ſtraight line which falls upon two other ſtraight lines makes the exterior angle equal to the interior and oppoſite upon the ſame ſide of the line; or makes the interior angles upon the ſame ſide together equal to two right angles; the two ſtraight lines ſhall be parallel to one another.

Let the ſtraight line EF which falls upon the two ſtraight lines AB, CD make the exterior angle EGB equal to the interior and oppoſite angle GHD upon the ſame ſide; or make the interior angles on the ſame ſide BGH, GHD together equal to two right angles. AB ſhall be parallel to CD.



Becauſe the angle EGB is equal to the angle GHD, and the angle EGB equal [a] to the angle AGH, a. 15. 1. the angle AGH ſhall be equal to the angle GHD; and they are the alternate angles; therefor AB is parallel [b] to CD. again, because the angles BGH, GHD are equal [c] to two right angles, and that AGH, BGH are alſo equal [d] to two right angles; the angles AGH, BGH ſhall be equal to the angles BGH, GHD. take away the common angle BGH, and the remaining angle AGH ſhall be equal to the remaining angle GHD; and they are alternate angles; therefor AB is parallel to CD. wherefor if a ſtraight line &c. Q. E. D.

b. 27. 1.
c. By Hypoth.
d. 13. 1.

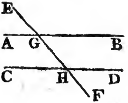
PROP. XXIX. THEOR.

See the Notes on this Prop.

IF a ſtraight line falls upon two parallel ſtraight lines, it ſhall make the alternate angles equal to one another; and the exterior angle equal to the interior and oppoſite upon the ſame ſide; and likewiſe it ſhall make the two interior angles upon the ſame ſide together equal to two right angles.

Let the ſtraight line EF fall upon the parallel ſtraight lines AB, CD. the alternate angles AGH, GHD ſhall be equal to one another; and the exterior angle EGB equal to the interior and oppoſite, upon the ſame ſide, GHD; and the two interior angles BGH, GHD upon the ſame ſide ſhall be together equal to two right angles.

For if AGH be not equal to GHD, one of them muſt be greater than the other; let AGH be the greater. and becauſe the angle AGH is greater than the angle GHD, add to each of them the angle BGH, and the angles AGH, BGH ſhall be greater than the angles BGH, GHD. but the angles AGH, BGH are equal [a] to two right angles; therefor the angles BGH, GHD are



a. 13. 1.

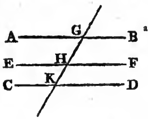
are leſſer than two right angles. but theſe ſtraight lines which with another ſtraight line falling upon them make the interior angles on the ſame ſide leſs than two right angles, ſhall meet * together if continually produced; therefor the ſtraight lines AB, CD if produced far enough ſhall meet. but they never meet, ſince they are parallel by the Hypotheſis. therefor the angle AGH is not unequal to the angle GHD, that is, it is equal to it. but the angle AGH is equal [b] to the angle EGB; therefor likewiſe EGB is equal to GHD. add to each of theſe the angle BGH, and the angles EGB, BGH, ſhall be equal to the angles BGH, GHD; but EGB, BGH are equal [c] to two right angles; therefor alſo BGH, GHD are equal to two right angles. wherefor if a ſtraight line &c. Q. E. D.

* 12. Ax. See the Notes on this Propoſition.
b. 15. 1.
c. 13. 1.

PROP. XXX. THEOR.

THESE ſtraight lines which are parallel to the ſame ſtraight line, are parallel to one another.

Let AB, CD be each of them parallel to EF; AB ſhall be parallel to CD.



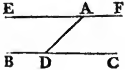
Let the ſtraight line GHK cut AB, EF, CD; and becauſe GHK cuts the parallel ſtraight lines AB, EF, the angle AGH is equal [a] to the angle GHF. again, becauſe the ſtraight line GK cuts the parallel ſtraight lines EF, CD, the angle GHF is equal [a] to the angle GKD. and it was ſhewn that the angle AGK is equal to the angle GHF; therefor alſo AGK is equal to GKD. and they are alternate angles; therefor AB is parallel [b] to CD. wherefor theſe ſtraight lines &c. Q. E. D.

a. 29. 1.
b. 27. 1.

PROP. XXXI. THEOR.

TO draw a ſtraight line thro' a given point parallel to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw a ſtraight line thro' the point A, parallel to the ſtraight line BC.



In BC take any point D, and join AD; and at the point A in the ſtraight line AD
a. 23. 1. make [a] the angle DAE equal to the angle ADC; and produce the ſtraight line EA to F.

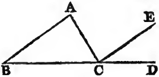
Becauſe therefor the ſtraight line AD which meets the two ſtraight lines BC, EF, makes the alternate angles EAD, ADC equal to one
b. 27. 1. another, EF ſhall be parallel [b] to BC. therefor the ſtraight line EAF is drawn thro' the given point A parallel to the given ſtraight line BC. Which was to be done.

PROP. XXXII. THEOR.

IF a ſide of any triangle be produced, the exterior angle ſhall be equal to the two interior and oppoſite angles; and the three interior angles of every triangle are equal to two right angles.

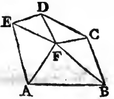
Let ABC be a triangle, and let one of its ſides BC be produced to D. the exterior angle ACD ſhall be equal to the two interior and oppoſite angles CAB, ABC; and the three interior angles of the triangle, viz. ABC, BCA, CAB ſhall together be equal to two right angles.

Thro' the point C draw CE parallel [a] to the ſtraight line AB. and a. 31. 1. becauſe AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE ſhall be equal [b] to one another. b. 29. 1. again becauſe AB is parallel to CE, and BD falls upon them, the exterior angle ECD ſhall be equal to the interior and oppoſite angle ABC. but the angle ACE was ſhewn to be equal to the angle BAC; therefor the whole exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC. to theſe equals add the angle ACB, and the angles ACD, ACB ſhall be equal to the three angles CBA, BAC, ACB. but the angles ACD, ACB are equal [c] to c. 13. 1. two right angles; therefor alſo the angles CBA, BAC, ACB are equal to two right angles. wherefor if a ſide of a triangle &c. Q. E. D.

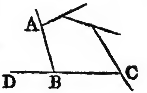


Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has ſides, by drawing ſtraight lines from a point F within the figure to each of its angles. And, by the preceeding Propoſition, all the angles of theſe triangles are equal to twice as many right angles as there are triangles, that is, as there are ſides of the figure. and the ſame angles are equal to the angles of the figure, together with the angles at the point F which is the common Vertex of the triangles; that is[a], together with four right angles. a. 2. Cor. 15. 1. Therefor all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.



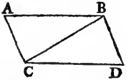
Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.



Becauſe every interior angle ABC with
b. 13. 1. its adjacent exterior ABD is equal [b] to two right angles, therefor all the interior together with all the exterior angles of the figure, are equal to twice as many right angles as there are ſides of the figure, that is, by the foregoing Corollary, they are equal to all the interior angles of the figure, together with four right angles. therefor all the exterior angles are equal to four right angles.

PROP. XXXIII. THEOR.

THE ſtraight lines which join the extremities of two equal and parallel ſtraight lines, towards the ſame parts, are alſo themſelves equal and parallel.



Let AB, CD be equal and parallel ſtraight lines, and joined towards the ſame parts by the ſtraight lines AC, BD; AC, BD ſhall be equal and parallel.

Join BC, and becauſe AB is parallel to CD, and BC meets them; the alternate angles
a. 29. 1. ABC, BCD are equal [a]; and becauſe AB is equal to CD, and BC common to the two triangles ABC, DCB, the two ſides AB, BC are equal to the two DC, CB; and the angle
b. 4. 1. ABC is equal to the angle BCD; therefor the baſe AC is equal [b] to the baſe BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles [b], each to each, to which the equal ſides are oppoſite. therefor the angle ACB is equal to the angle CBD. and becauſe

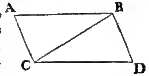
becauſe the ſtraight line BC meets the two ſtraight lines AC, BD and makes the alternate angles ACB, CBD equal to one another, AC is parallel [c] to BD. and it was ſhewn to be equal to it. therefor ſtraight lines &c. Q. E. D. c. 27. 1.

PROP. XXXIV. THEOR.

THE oppoſite ſides and angles of parallelograms are equal to one another, and the diameter biſſects them, that is, divides them into two equal parts.

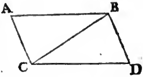
N. B. *A Parallelogram is a four ſided figure whoſe oppoſite ſides are parallel. and the diameter is the ſtraight line joining two of its oppoſite angles.*

Let ABCD be a parallelogram, whoſe diameter is BC. the oppoſite ſides and angles of it ſhall be equal to one another; and the diameter BC ſhall biſſect it.



Becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD ſhall be equal [a] to one a. 29. 1. another, and becauſe AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal [a] to one another. wherefor the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD in the other, each to each, and one ſide BC common to the two triangles, which is adjacent to their equal angles; therefor their other ſides ſhall be equal, each to each, and the third angle of the one to the third angle of the other[b], viz. the ſide AB to b. 26. 1. the ſide CD, and AC to BD, and the angle BAC equal to the angle BDC. and becauſe the angle ABC is equal to the angle BCD, and the

angle CBD to the angle ACB; the whole angle ABD ſhall be equal to the whole angle ACD. and the angle BAC has been ſhewn to be equal to the angle BDC; therefor the oppoſite ſides and angles of parallelograms are equal to one another. and beſides, their diameter ſhall biſſect them. for, becauſe AB is equal to CD, and BC common; the two AB, BC are equal to the two DC, CB, each to each; and the angle ABC is equal to the angle BCD; there-
c. 4. 1. for the triangle ABC is equal [c] to the triangle BCD, and the diameter BC divides the parallelogram ACDB into two equal parts. Q. E. D.

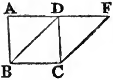


PROP. XXXV. THEOR.

PARALLELOGRAMS upon the ſame baſe and between the ſame parallels, are equal to one another.

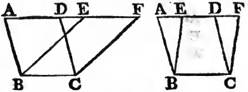
See the 2d and 3d Figures. Let the parallelograms ABCD, EBCF be upon the ſame baſe BC and between the ſame parallels AF, BC. the parallelogram ABCD ſhall be equal to the parallelogram EBCF.

If the ſides AD, DF of the parallelograms ABCD, DBCF oppoſite to the baſe BC, be terminated in the ſame point D; it is plain that each of the parallelo-
a. 34. 1. grams is double [a] of the triangle BDC; and are therefor equal to one another.



But if the ſides AD, EF oppoſite to the baſe BC of the parallelograms ABCD, EBCF be not terminated in the ſame point; then becauſe ABCD is a parallelogram, AD is equal [a] to BC; by the ſame reaſon, EF is equal to BC;
b. 1. Ax. wherefor AD is equal [b] to EF; and DE is common; therefor the

whole, or the remainder, AE is equal [c] to the whole, or the remainder
DF; AB alſo is equal to DC; and the two EA, AB are therefor e- c. 2. or 3. Ax.

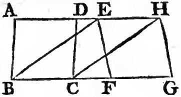


qual to the two FD, DC, each to each; and the exterior angle FDC
is equal [d] to the interior EAB; therefor the baſe EB is equal to the d. 29. 1.
baſe FC, and the triangle EAB equal [e] to the triangle FDC. take the e. 4. 1.
triangle FDC from the trapezium ABCF, and from the ſame trapezium
take the triangle EAB, the remainders ſhall be equal[f], that is, the f. 3. Ax.
parallelogram ABCD ſhall be equal to the parallelogram EBCF.
therefor parallelograms upon the ſame baſe &c. Q. E. D.

PROP. XXXVI. THEOR.

PARALLELOGRAMS upon equal baſes and between the ſame parallels, are equal to one another.

Let ABCD, EFGH be parallelograms upon equal baſes BC, FG, and between the ſame parallels AH, BG; the parallelogram ABCD ſhall be equal to EFGH.

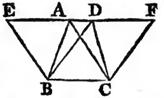


Join BE, CH; and becauſe BC is equal to FG, and FG to [a] EH, BC a. 34. 1.
ſhall be equal to EH; and they are parallels, and joined towards the ſame parts by the ſtraight lines BE, CH. but ſtraight lines which join equal and parallel ſtraight lines to-

wards the ſame parts, are themſelves equal and parallel[b]; therefor EB,
b. 33. 1. CH are both equal and parallel and EBCH a parallelogram; and it
c. 35. 1. is equal [c] to ABCD, becauſe it is upon the ſame baſe BC, and between the ſame parallels BC, AD. by the like reaſon the parallelogram EFGH is equal to the ſame EBCH. therefor alſo the parallelogram ABCD is equal to EFGH. Wherefor parallelograms &c. Q. E. D.

PROP. XXXVII. THEOR.

TRIANGLES upon the ſame baſe, and between the ſame parallels, are equal to one another.



Let the triangles ABC, DBC be upon the ſame baſe BC and between the ſame parallels AD, BC. the triangle ABC ſhall be equal to the triangle DBC.

Produce AD both ways to the points
a. 31. 1. E, F, and thro' B draw [a] BE parallel to CA; and thro' C draw CF parallel to BD. therefor each of the figures EBCA, DBCF is a parallelogram;
b. 35. 1. and EBCA is equal [b] to DBCF, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, EF; and of the parallelogram EBCA the triangle ABC is the half, becauſe the diameter AB
c. 34. 1. biſſects [c] it; and the triangle DBC is the half of the parallelogram DBCF, becauſe the diameter DC biſſects it. but the halves of equal
d. 7. Ax. things are equal[d]; therefor the triangle ABC is equal to the triangle DBC. Wherefor triangles &c. Q. E. D.

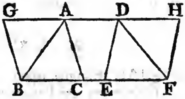
PROP. XXXVIII.

PROP. XXXVIII. THEOR.

TRIANGLES upon equal baſes, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal baſes BC, EF, and between the ſame parallels BF, AD. the triangle ABC ſhall be equal to the triangle DEF.

Produce AD both ways to the points G, H, and thro' B draw BG parallel [a] to CA, and thro' F draw FH parallel to ED. then each of a. 31. 1.
the figures GBCA, DEFH is a parallelogram; and they are equal [b] to b. 36. 1.
one another, becauſe they are upon equal baſes BC, EF and between the ſame parallels BF, GH; and the triangle ABC is the half [c] of c. 34. 1.
the parallelogram GBCA, becauſe the diameter AB biſſects it; and of the parallelogram DEFH the triangle DEF is the half [c], becauſe the diameter DF biſſects it. but the halves of equal things are equal [d]; d. 7. Ax.
therefor the triangle ABC is equal to the triangle DEF. Wherefor triangles &c. Q. E. D.

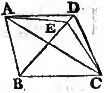


PROP. XXXIX. THEOR.

EQUAL triangles upon the ſame baſe, and upon the ſame ſide of it, are between the ſame parallels.

Let the equal triangles ABC, DBC be upon the ſame baſe BC, and upon the ſame ſide of it; they ſhall be between the ſame parallels.

Join AD; AD ſhall be parallel to BC; for if it is not, thro' the
a. 31. 1. point A draw [a] AE parallel to BC, and join EC.
b. 37. 1. the triangle ABC is equal [b] to the triangle EBC, becauſe it is upon the ſame baſe BC, and between the ſame parallels BC, AE. but the triangle ABC is equal to the triangle BDC; therefor alſo the triangle BDC is equal to the triangle EBC, the greater to the leſs, which is impoſſible. therefor AE is not parallel to BC. in the ſame manner it ſhall be demonſtrated that no other line but AD is parallel to BC; AD is therefor parallel to it. Wherefor equal triangles upon &c. Q. E. D.

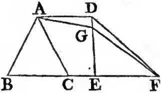


PROP. XL. THEOR.

EQUAL triangles upon equal baſes, and towards the ſame parts, are between the ſame parallels.

Let the equal triangles ABC, DEF be upon equal baſes BC, EF, and towards the ſame parts; they ſhall be between the ſame parallels.

Join AD, it ſhall be parallel to BC.
a. 31. 1. for if it is not, thro' A draw [a] AG parallel to BF, and join GF. the triangle
b. 38. 1. ABC is equal [b] to the triangle GEF, becauſe they are upon equal baſes BC, EF, and between the ſame parallels BF, AG. but the triangle ABC is equal to the triangle DEF; therefor alſo the triangle DEF ſhall be equal to the triangle GEF, the greater to the leſs, which is impoſſible. therefor AG is not parallel to BF. and in the ſame manner it ſhall be demonſtrated that there

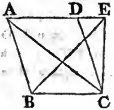


there is no other parallel to it but AD, AD is therefor parallel to BF. Wherefor equal triangles &c. Q. E. D.

PROP. XLI. THEOR.

If a parallelogram and triangle be upon the ſame baſe, and between the ſame parallels; the parallelogram ſhall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the ſame baſe BC, and between the ſame parallels BC, AE; the parallelogram ABCD ſhall be double of the triangle EBC.



Join AC; then the triangle ABC is equal [a] to the triangle EBC, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, AE. but the parallelogram ABCD is double [b] of the triangle ABC, becauſe the diameter AC divides it into two equal parts; wherefor ABCD ſhall alſo be double of the triangle EBC. therefor if a parallelogram &c. Q. E. D.

a. 37. 1.

b. 34. 1.

PROP. XLII. PROB.

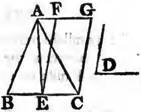
To deſcribe a parallelogram that ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle. It is required to deſcribe a parallelogram that ſhall be equal to the given triangle ABC, and have one of its angles equal to D.

Biſſect [a] BC in E, join AE, and at the point E in the ſtraight line

a. 10. 1.

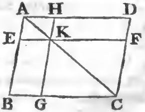
line EC make [b] the angle CEF equal to D; and thro' A draw
b. 23. 1. [c] AG parallel to EC, and thro' C draw [c] CG parallel to EF. there-
c. 31. 1. for FECG is a parallelogram. and becauſe BE is equal to EC, the triangle ABE ſhall
d. 38. 1. be equal [d] to the triangle AEC, becauſe they are upon equal baſes BE, EC and between the ſame parallels BC, AG; therefor the triangle ABC is double of the triangle AEC. and the parallelogram FECG
e. 41. 1. is likewiſe double [e] of the triangle AEC, becauſe it is upon the ſame baſe, and between the ſame parallels. therefor the parallelogram FECG is equal to the triangle ABC, and it has one of its angles CEF equal to the given angle D. wherefor there has been deſcribed a parallelogram FECG equal to a given triangle ABC, having one of its angles CEF equal to the given angle D. Which was to be done.



PROP. XLIII. THEOR.

THE complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, whoſe diameter is AC, and EH, FG the parallelograms about AC, *that is, thro' which* AC *paſſes*, and BK, KD the other parallelograms which make up the whole figure ABCD, which are therefor called the complements. the complement BK ſhall be equal to the complement KD.



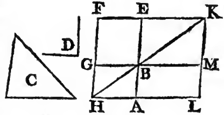
Becauſe ABCD is a parallelogram, and AC its diameter, the triangle ABC is equal [a] to the triangle ADC. and becauſe EKHA is a parallelogram, whoſe diameter is AK, the triangle AEK is equal to the triangle AHK. by the ſame reaſon, the triangle KGC is equal to the triangle KFC. therefor becauſe the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK together with the triangle KGC ſhall be equal to the triangle AHK together with the triangle KFC. but the whole triangle ABC is equal to the whole ADC; therefor the remaining complement BK is equal to the remaining complement KD. Wherefor the complements &c. Q. E. D.

a. 34. 1.

PROP. XLIV. PROB.

TO apply a parallelogram to a given ſtraight line, which ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given ſtraight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the ſtraight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

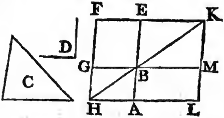


Make [a] the parallelogram BEFG equal to the triangle C and having the angle EBG equal to the angle D, ſo that BE be in the ſame ſtraight line with AB, and produce FG to H; and thro' A draw [b] AH parallel to BG or EF, and join HB. then becauſe the ſtraight line HF falls upon the parallels AH, EF, the angles AHF, HFE

a. 42. 1.

b. 31. 1.

HFE are together equal [c] to two right angles; wherefor the angles
c. 29. 1. BHF, HFE are leſſer than two right angles. but ſtraight lines which
with an other ſtraight line make the interior angles upon the ſame ſide
d. 12. Ax. leſs than two right angles, do meet [d] if produced far enough. therefor
HB, FE ſhall meet, if produced; let them meet in K, and thro' K
draw KL parallel to EA or FH, and produce HA, GB to the points
L, M. then HLKF is a parallelogram, whoſe diameter is HK, and
AG, ME are the parallelograms about HK; and LB, BF are the
complements; therefor LB is
e. 43. 1. equal [e] to BF. but BF is equal to the triangle C; wherefor LB is equal to the triangle
C. and becauſe the angle GBE
f. 15. 1. is equal [f] to the angle ABM,
and likewiſe to the angle D;
the angle ABM ſhall be equal to the angle D. therefor the parallelogram LB is applied to the ſtraight line AB equal to the triangle C,
and having the angle ABM equal to the angle D. Which was to be
done.

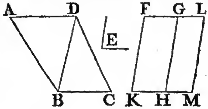


PROP. XLV. PROB.

TO deſcribe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to deſcribe a parallelogram equal to ABCD and having an angle equal to E.

a. 42. 1. Join DB, and deſcribe [a] the parallelogram FH equal to the triangle ADB,



ADB, and having the angle HKF equal to the angle E; and to the ſtraight line GH apply [b] the parallelogram GM equal to the triangle b. 44. 1. DBC having the angle GHM equal to the angle E. and becauſe the angle E is equal to each of the angles FKH, GHM, the angle FKH ſhall be equal to GHM; add to each of theſe the angle KHG, and the angles FKH, KHG ſhall be equal to the angles KHG, GHM. but FKH, KHG are equal [c] to two right angles; therefor alſo KHG, GHM are c. 29. 1. equal to two right angles. and becauſe at the point H in the ſtraight line GH, the two ſtraight lines KH, HM upon the oppoſite ſides of it make the adjacent angles equal to two right angles, KH ſhall be in the ſame ſtraight [d] line with HM. and becauſe the ſtraight line HG d. 14. 1. meets the parallels KM, FG, the alternate angles MHG, HGF are equal [c]; add to each of theſe the angle HGL; and the angles MHG, HGL ſhall be equal to the angles HGF, HGL. but the angles MHG, HGL are equal [c] to two right angles; wherefor alſo the angles HGF, HGL are equal to two right angles, and FG ſhall be in the ſame ſtraight line with GL. and becauſe KF is parallel to HG, and HG to ML; KF ſhall be parallel [e] to ML. and KM, FL are parallels; where- e. 30. 1. for KFLM is a parallelogram. and becauſe the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM; the whole rectilineal figure ABCD ſhall be equal to the whole parallelogram KFLM. therefor the parallelogram KFLM has been deſcribed equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E. Which was to be done.

Cor. From this it is manifeſt how to apply a parallelogram to a

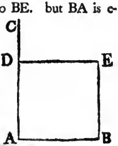
b. 44. 1. given ſtraight line, having an angle equal to a given rectilineal angle, that ſhall be equal to a given rectilineal figure, viz. by applying [b] to the given ſtraight line, a parallelogram equal to the firſt triangle ABD, having an angle equal to the given angle.

PROP. XLVI. PROB.

TO deſcribe a ſquare upon a given ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe a ſquare upon AB.

a. 11. 1. b. 3. 1. c. 31. 1. From the point A draw [a] AC at right angles to AB; and make [b] AD equal to AB, and thro' the point D draw DE parallel [c] to it, and thro' B draw BE parallel to AD. therefor ADEB is a parallelo-
d. 34. 1. gram; whence AB is equal [d] to DE, and AD to BE. but BA is equal to AD; therefor the four ſtraight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral. likewiſe all its angles are right angles; for becauſe the ſtraight line AD meets the parallels AB, DE,
e. 29. 1. the angles BAD, ADE are equal [e] to two right angles; but BAD is a right angle, therefor alſo ADE is a right angle. but the oppoſite angles of parallelograms are equal [d]; therefor each of the oppoſite angles ABE, BED is a right angle; wherefor the figure ADEB is rectangular. and it has been demonſtrated that it is equilateral; it is therefor a ſquare, and it is deſcribed upon the given ſtraight line AB. Which was to be done.



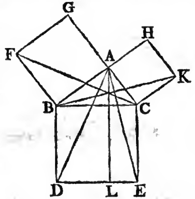
Cor. Hence every parallelogram that has one right angle has all its angles right angles.

PROP. XLVII.

PROP. XLVII. THEOR.

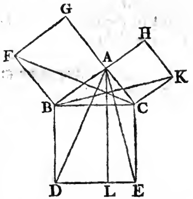
IN any right angled triangle, the ſquare which is deſcribed upon the ſide ſubtending the right angle, is equal to the ſquares deſcribed upon the ſides which contain the right angle.

Let ABC be a right angled triangle having the right angle BAC; the ſquare deſcribed upon the ſide BC, ſhall be equal to the ſquares deſcribed upon BA, AC.



On BC deſcribe [a] the ſquare BDEC, and on BA, AC the ſquares a. 46. 1.
GB, HC; and thro' A draw [b] AL parallel to BD or CE, and join AD, b. 31. 1.
FC. therefor becauſe each of the
angles BAC, BAG is a right angle[c], c. 30. Def.
the two ſtraight lines AC, AG upon the oppoſite ſides of AB, make
with it at the point A the adjacent
angles equal to two right angles;
therefor CA is in the ſame ſtraight
line [d] with AG. by the ſame rea- d. 14. 1.
ſon, AB and AH are in the ſame
ſtraight line. and becauſe the angle
DBC is equal to the angle FBA,
for each of them is a right angle,
add to each of them the angle ABC, and the whole angle DBA ſhall be
equal [e] to the whole FBC. and becauſe the two ſides AB, BD are equal e. 2. Ax.
to the two FB, BC, each to each, and the angle DBA equal to the angle
FBC; the baſe AD ſhall be equal [f] to the baſe FC, and the triangle f. 4. 1.
ABC to the triangle FBC. now the parallelogram BL is double [g] of g. 41. 1.

the triangle ABD, becauſe they are upon the ſame baſe BD, and between the ſame parallels BD, AL; and the ſquare GB is double of the triangle FBC, becauſe theſe alſo are upon the ſame baſe FB, and between the ſame parallels FB, GC. but the doubles of equals are e-h. 6. Ax. qual [h] to one another. therefor the parallelogram BL is equal to the ſquare GB. and in the ſame manner, by joining AE, BK, it ſhall be demonſtrated that the parallelogram CL is equal to the ſquare HC. Therefor the whole ſquare BDEC is equal to the two ſquares GB, HC. and the ſquare BDEC is deſcribed upon the ſtraight line BC, and the ſquares GB, HC upon BA, AC. wherefor the ſquare upon the ſide BC is equal to the ſquares upon the ſides BA, AC. Therefor in any right angled triangle &c. Q. E. D.

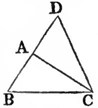


PROP. XLVIII. THEOR.

IF the ſquare deſcribed upon one of the ſides of a triangle, be equal to the ſquares deſcribed upon the other two ſides of it; the angle contained by theſe two ſides ſhall be a right angle.

If the ſquare deſcribed upon BC one of the ſides of the triangle ABC be equal to the ſquares upon the other ſides BA, AC. the angle BAC ſhall be a right angle.

a. 11. 1. From the point A draw [a] AD at right angles to AC, and make AD equal



equal to BA, and join DC. therefor becauſe DA is equal to AB, the ſquare of DA ſhall be equal to the ſquare of AB; to each of theſe add the ſquare of AC, and the ſquares of DA, AC ſhall be equal to the ſquares of BA, AC. but the ſquare of DC is equal [b] to the ſquares of DA, AC, becauſe DAC is a right angle; and the ſquare of BC, by Hypotheſis, is equal to the ſquares of BA, AC; therefor the ſquare of DC is equal to the ſquare of BC; and therefor alſo the ſide DC is equal to the ſide BC. and becauſe the ſide DA is equal to AB, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC; and the baſe DC is equal to the baſe BC; therefor the angle DAC is equal [c] to the angle BAC. but DAC is a right angle, therefor alſo BAC is a right angle. Therefor if the ſquare &c. Q. E. D.

b. 47. 1.

c. 8. 1.

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BOOK II.

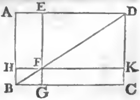
DEFINITIONS.

I.

EVERY right angled parallelogram is ſaid to be contained by any two of the ſtraight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about the diameter, together with the two complements, is called the Gnomon. ‘Thus the parallelogram HG together with the complements AF, FC is the gnomon, which is more briefly expreſſed by the letters AGK, or EHC which are at the oppoſite angles of the parallelograms which make the gnomon.’

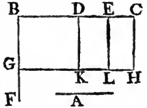


PROP. I.

PROP. I. THEOR.

IF there be two ſtraight lines, one of which is divided into any number of parts; the rectangle contained by the two ſtraight lines, is equal to the rectangles contained by the undivided line, and the ſeveral parts of the divided line.

Let A and BC be two ſtraight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the ſtraight lines A, BC ſhall be equal to the rectangle contained by A, BD; to that contained by A, DE; and alſo to that contained by A, EC.



From the point B draw [a] BF at right angles to BC, and make BG equal [b] to A; and thro' G draw [c] GH parallel to BC; and thro' D, E, C draw [c] DK, EL, CH parallel to BG. then the rectangle BH is equal to the rectangles BK, DL, EH; and BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, becauſe DK, that is [d] BG, is equal to A; and in like manner the rectangle EH is contained by A, EC. therefor the rectangle contained by A, BC is equal to the ſeveral rectangles contained by A, BD, and by A, DE, and alſo by A, EC. Wherefor if there be two ſtraight lines &c. Q. E. D.

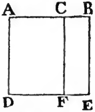
a. 11. 1.
b. 3. 1.
c. 31. 1.
d. 34. 1.

PROP. II.

PROP. II. THEOR.

IF a ſtraight line be divided into any two parts, the rectangles contained by the whole and each of the parts, is equal to the ſquare of the whole line.

Let the ſtraight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC together with the rectangle * AB, AC ſhall be equal to the ſquare of AB.



a. 46. 1. Upon AB deſcribe [a] the ſquare ADEB, and
b. 31. 1. thro' C draw [b] CF parallel to AD or BE. then AE is equal to the rectangles AF, CE; and AE is the ſquare of AB; and AF is the rectangle contained by BA, AC; for it is contained by DA, AC of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB. therefor the rectangle contained by AB, AC together with the rectangle AB, BC, is equal to the ſquare of AB. If therefor a ſtraight line &c. Q. E. D.

PROP. III. THEOR.

IF a ſtraight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the ſquare of the foreſaid part.

Let the ſtraight line AB be divided into any two parts in the point C;

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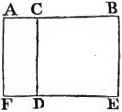
* N. B. To avoid repeating the word *Contained* too frequently, the rectangle contained by two ſtraight lines AB, AC is ſometimes ſimply called the rectangle AB, AC.

the rectangle AB, BC ſhall be equal to the rectangle AC, CB together with the ſquare of BC.

Upon BC deſcribe [a] the ſquare CDEB, and produce ED to F, and thro' A draw [b] AF parallel to CD or BE. then the rectangle AE is equal to the rectangles AD, CE; and AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; and DB is the ſquare of BC. therefor the rectangle AB, BC is equal to the rectangle AC, CB together with the ſquare of BC. if therefor a ſtraight line &c. Q. E. D.

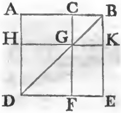
a. 46. 1.
b. 31. 1.



PROP. IV. THEOR.

IF a ſtraight line be divided into any two parts, the ſquare of the whole line is equal to the ſquares of the two parts, together with twice the rectangle contained by the parts.

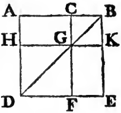
Let the ſtraight line AB be divided into any two parts in C; the ſquare of AB ſhall be equal to the ſquares of AC, CB and to twice the rectangle contained by AC, CB.



Upon AB deſcribe [a] the ſquare ADEB, and join BD, and thro' C draw [b] CGF parallel to AD or BE, and thro' G draw HK parallel to AB or DE. and becauſe CF is parallel to AD, and BD falls upon them, the exterior angle BGC ſhall be equal [c] to the interior and oppoſite angle ADB; but

a. 46. 1.
b. 31. 1.
c. 29. 1.

ADB is equal [d] to the angle ABD, becauſe BA is equal to AD, being
d. 5. 1. ſides of a ſquare; wherefor the angle CGB is equal to the
e. 6. 1. angle GBC, and therefor the ſide BC is equal [e] to the ſide CG.
f. 34. 1. but alſo CB is equal [f] to GK, and CG to BK; wherefor the figure CGKB is equilateral. it is likewiſe rectangular; for becauſe CG is parallel to BK, and CB meets them, the angles KBC, GCB are equal to two right angles; and KBC is a right angle, wherefor GCB is a right angle; and therefor alſo the angles [f] CGK, GKB oppoſite to theſe are right angles, and CGKB is rectangular. but it is alſo equilateral, as was demonſtrated; wherefor it is a ſquare, and it is upon the ſide CB. by the ſame reaſon HF alſo is a ſquare, and it is upon the ſide HG which is equal to AC. therefor HF, CK are the ſquares of AC, CB. and
g. 43. 1. becauſe the complement AG is equal [g] to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB; GE ſhall alſo be equal to the rectangle AC, CB; wherefor AG, GE are equal to twice the rectangle AC, CB. and HF, CK are the ſquares of AC, CB; wherefor the four figures HF, CK, AG, GE are equal to the ſquares of AC, CB and to twice the rectangle AC, CB. but HF, CK, AG, GE make up the whole figure ADEB which is the ſquare of AB. therefor the ſquare of AB is equal to the ſquares of AC, CB and twice the rectangle AC, CB. wherefor if a ſtraight line &c. Q. E. D.



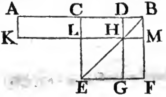
Cor. From the demonſtration it is manifeſt, that the parallelograms about the diameter of a ſquare are likewiſe ſquares.

PROP. V.

PROP. V. THEOR.

IF a ſtraight line be divided into two equal parts, and alſo into two unequal parts; the rectangle contained by the unequal parts, together with the ſquare of the line between the points of ſection, is equal to the ſquare of half the line.

Let the ſtraight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB together with the ſquare of CD, ſhall be equal to the ſquare of CB.



Upon CB deſcribe [a] the ſquare CEFB, join BE, and thro' D draw [b] DHG parallel to CE or BF; and thro' H draw KLM parallel to CB or EF; and alſo thro' A draw AK parallel to CL or BM. and becauſe the complement CH is equal [c] to the complement HF, to each of theſe add DM, and the whole CM ſhall be equal to the whole DF; but CM is equal [d] to AL, becauſe AC is equal to CB; therefor alſo AL is equal to DF. to each of theſe add CH, and the whole AH ſhall be equal to DF and CH. but AH is the rectangle contained by AD, DB, for DH is equal [e] to DB; and DF together with CH is the gnomon CMG; therefor the gnomon CMG is equal to the rectangle AD, DB. to each of thoſe add LG, which is equal [e] to the ſquare of CD, and the gnomon CMG together with LG ſhall be equal to the rectangle AD, DB together with the ſquare of CD. but the gnomon CMG and LG make up the whole figure CEFB, which is the ſquare of CB. therefor the rectangle AD, DB together with

a. 46. 1.
b. 31. 1.
c. 43. 1.
d. 36. 1.
e. Cor. 4. 2.

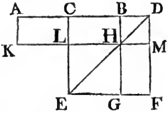
the ſquare of CD is equal to the ſquare of CB. wherefor if a ſtraight line &c. Q. E. D.

PROP. VI. THEOR.

IF a ſtraight line be biſſected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the ſquare of half of the line biſſected, is equal to the ſquare of the ſtraight line which is made up of the half and the part produced.

Let the ſtraight line AB be biſſected in C, and produced to the point D; the rectangle AD, DB together with the ſquare of CB, ſhall be equal to the ſquare of CD.

a. 46. 1. Upon CD deſcribe [a] the ſquare CEFD, join DE, and thro' B draw
b. 31. 1. [b] BHG parallel to CE or DF, and thro' H draw KLM parallel to AD
or EF, and alſo thro' A draw AK parallel to CL or DM. and becauſe
c. 36. 1. AC is equal to CB, the rectangle AL ſhall be equal [c] to CH; but CH
d. 43. 1. is equal [d] to HF; therefor alſo AL is equal to HF. to each of theſe add CM, and the whole AM ſhall be equal to the gnomon CMG. and AM is the rectangle contained by AD, DB, for
e. Cor. 4. 2. DM is equal [e] to DB. therefor the gnomon CMG is equal to the rectangle AD, DB. add to each of theſe LG which is equal to the ſquare of CB; and the rectangle AD, DB together with the ſquare of CB ſhall be equal to the gnomon CMG and the figure LG. but the gnomon CMG and LG make up the whole figure CEFD, which is the ſquare of CD;



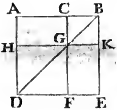
CD; therefor the rectangle AD, DB together with the ſquare of CB, is equal to the ſquare of CD. wherefor if a ſtraight line &c. Q. E. D.

PROP. VII. THEOR.

IF a ſtraight line be divided into any two parts, the ſquares of the whole line, and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with the ſquare of the other part.

Let the ſtraight line AB be divided into any two parts in the point C; the ſquares of AB, BC ſhall be equal to twice the rectangle AB, BC together with the ſquare of AC.

Upon AB deſcribe [a] the ſquare ADEB, and conſtruct the figure as in the preceeding Propoſitions. and becauſe AG is equal [b] to GE, add to each of them CK, and the whole AK ſhall be equal to the whole CE; therefor AK, CE are double of AK. but AK, CE are the gnomon AKF together with the ſquare CK; therefor the gnomon AKF together with the ſquare CK is double of AK. but twice the rectangle AB, BC is double of AK, for BK is equal [c] to BC. therefor the gnomon AKF together with the ſquare CK is equal to twice the rectangle AB, BC. to each of theſe equals add HF, which is equal to the ſquare of AC; and the gnomon AKF together with the ſquares CK, HF ſhall be equal to twice the rectangle AB, BC and the ſquare of AC. but the gnomon AKF together with the ſquares CK, HF make up the whole figure ADEB and CK, which are the ſquares of AB and BC. therefor the ſquares of



a. 46. 1.
b. 43. 1.
c. Cor. 4. 2.

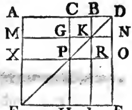
AB and BC are equal to twice the rectangle AB, BC together with the ſquare of AC. wherefor if a ſtraight line &c. Q. E. D.

PROP. VIII. THEOR.

If a ſtraight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the ſquare of the other part, is equal to the ſquare of the ſtraight line which is made up of the whole and that part.

Let the ſtraight line AB be divided into any two parts in the point C; four times the rectangle AB, BC, together with the ſquare of AC, ſhall be equal to the ſquare of the ſtraight line made up of AB and BC together.

Produce AB to D ſo that BD be equal to CB, and upon AD deſcribe the ſquare AEFD; and conſtruct two figures ſuch as in the preceeding. Becauſe CB is equal to BD, and



a. 34. 1. that CB is equal [a] to GK, and BD to KN; GK ſhall be equal to KN. by the ſame reaſon PR is equal to RO. and becauſe CB is equal to BD, and GK to KN, the rectangle CK
b. 36. 1. ſhall be equal [b] to BN, and GR to RN. but
c. 43. 1. CK is equal [c] to RN, becauſe they are the complements of the parallelogram CO; therefor alſo BN is equal to GR. and the four rectangles BN, CK, GR, RN, are therefor equal to one another, and ſo are quadruple of one of them CK. again becauſe CB is equal to BD,
d. Cor. 2. 4. and that BD is equal [d] to BK, that is to CG; and CB equal to GK, that [d] is to GP; CG ſhall be equal to GP. and becauſe CG is equal

qual to GP, and PR to RO, the rectangle AG ſhall be equal to MP, and PL to RF. but MP is equal [c] to PL, becauſe they are the complements of the parallelogram ML; wherefor AG alſo is equal to RF. therefor the four rectangles AG, MP, PL, RF are equal to one another, and ſo are quadruple of one of them AG. And it was demonſtrated that the four CK, BN, GR, RN are quadruple of CK. therefor the eight rectangles which contain the gnomon AOH, are quadruple of AK. and becauſe AK is the rectangle contained by AB, BC, for BK is equal to BC; four times the rectangle AB, BC ſhall be quadruple of AK. but the gnomon AOH was demonſtrated to be quadruple of AK; therefor four times the rectangle AB, BC is equal to the gnomon AOH. to each of theſe add XH, which is equal [d] to the ſquare of AC; and four times the rectangle AB, BC together with the ſquare of AC ſhall be equal to the gnomon AOH and the ſquare XH. but the gnomon AOH and XH make up the figure AEFD which is the ſquare of AD. therefor four times the rectangle AB, BC together with the ſquare of AC is equal to the ſquare of AD, that is, of AB and BC added together in one ſtraight line. wherefor if a ſtraight line &c. Q. E. D.

[c] 43. 1.

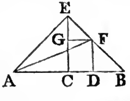
[d] Cor. 2. 4.

PROP. IX. THEOR.

IF a ſtraight line be divided into two equal parts, and alſo into two unequal; the ſquares of the two unequal parts, are together double of the ſquare of half the line, and of the ſquare of the line between the points of ſection.

Let the ſtraight line AB be divided into two equal parts in the point C, and into two unequal at the point D. the ſquares of AD, DB ſhall together be double of the ſquares of AC, CD.

From the point C draw [a] CE at right angles to AB, and make it
a. 11. 1. equal to AC or CB, and join EA, EB; thro' D draw [b] DF parallel
b. 31. 1. to CE, and thro' F draw FG parallel to AB; and join AF. then
c. 5. 1. becauſe AC is equal to CE, the angle EAC ſhall be equal [c] to the
angle AEC; and becauſe the angle ACE is a right angle, the two
d. 32. 1. others AEC, EAC ſhall together make one right angle[d]; and they are
equal to one another; each of them therefor is half of a right angle.
by the ſame reaſon each of the angles CEB, EBC is half of a right
angle; and therefor the whole AEB is a right angle. and becauſe the
angle GEF is half of a right angle, and EGF a right angle, for it is equal
e. 29. 1. [e] to the interior and oppoſite angle ECB, the
remaining angle EFG ſhall be half of a right
angle; therefor the angle GEF is equal to
f. 6. 1. the angle EFG, and the ſide EG equal [f] to
the ſide GF. again, becauſe the angle at B is
half of a right angle, and FDB a right angle,
for it is equal [e] to the interior and oppoſite angle ECB, the remaining
angle BFD ſhall be half of a right angle; therefor the angle at B is equal
to the angle BFD, and the ſide DF to [f] the ſide DB. and becauſe
AC is equal to CE, the ſquare of AC ſhall be equal to the ſquare of
CE; therefor the ſquares of AC, CE are double of the ſquare of AC.
g. 47. 1. but the ſquare of EA is equal [g] to the ſquares of AC, CE, becauſe ACE
is a right angle; therefor the ſquare of EA is double of the ſquare of
AC. again becauſe EG is equal to GF, the ſquare of EG ſhall be equal
to the ſquare of GF; therefor the ſquares of EG, GF are double
of the ſquare of GF; but the ſquare of EF is equal to the ſquares of
EG, GF; therefor the ſquare of EF is double of the ſquare of GF.
h. 34. 1. and GF is equal [h] to CD; therefor the ſquare of EF is double of the
ſquare of CD. but the ſquare of AE, is likewiſe double of the ſquare
of



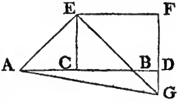
of AC; therefor the ſquares of AE, EF are double of the ſquares of AC, CD. and the ſquare of AF is equal [g] to the ſquares of AE, EF g. 47. 1. becauſe AEF is a right angle; therefor the ſquare of AF is double of the ſquares of AC, CD. but the ſquares of AD, DF are equal to the ſquare of AF, becauſe the angle ADF is a right angle; therefor the ſquares of AD, DF are double of the ſquares of AC, CD. and DF is equal to DB; therefor the ſquares of AD, DB are double of the ſquares of AC, CD. If therefor a ſtraight line &c. Q. E. D.

PROP. X. THEOR.

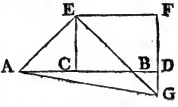
IF a ſtraight line be biſſected, and produced to any point, the ſquare of the whole line thus produced, and the ſquare of the part of it produced are together double of the ſquare of half of the line biſſected, and of the ſquare of the line made up of the half and the part produced.

Let the ſtraight line AB be biſſected in C, and produced to the point D; the ſquares of AD, DB ſhall be double of the ſquares of AC, CD.

From the point C draw [a] CE at right angles to AB, and make it a. 11. 1. equal to AC or CB, and join AE, EB; thro' E draw [b] EF parallel to b. 31. 1. AB, and thro' D draw DF parallel to CE. and becauſe the ſtraight line EF meets the parallels EC, FD, the angles CEF, EFD are equal [c] to two c. 29. 1. right angles; and therefor the angles BEF, EFD are leſs than two right angles. but ſtraight lines which with another ſtraight line make



the interior angles upon the ſame ſide leſs than two right angles, do
d. 12. Ax. meet [d] if produced far enough. therefor EB, FD ſhall meet, if produced,
toward BD. let them meet in G, and join AG. then becauſe
e. 5. 1. AC is equal to CE, the angle CEA ſhall be equal [e] to the angle EAC;
and the angle ACE is a right angle; therefor each of the angles CEA,
* 32. 1. EAC is half of a right angle*. by the ſame reaſon, each of the angles
CEB, EBC is half of a right angle; therefor AEB is a right angle.
f. 15. 1. and becauſe EBC is half of a right angle, DBG alſo ſhall [f] be half
of a right angle, for they are vertically oppoſite; but BDG is a right
c. 29. 1. angle, becauſe it is equal [c] to the alternate angle DCE; therefor the
remaining angle DGB is half of a
right angle, and is therefor equal to
the angle DBG; wherefor alſo the
g. 6. 1. ſide BD is equal [g] to the ſide DG.
again, becauſe EGF is half of a
right angle, and that the angle at F
h. 34. 1. is a right angle, becauſe it is equal [h] to the oppoſite angle ECD, the remaining
angle FEG ſhall be half of a right angle, and equal to the
angle EGF; wherefor alſo the ſide GF is equal [g] to the ſide FE. And
becauſe EC is equal to CA, the ſquare of EC ſhall be equal to the
ſquare of CA; therefor the ſquares of EC, CA are double of the
i. 47. 1. ſquare of CA. but the ſquare of EA is equal [i] to the ſquares of EC,
CA; therefor the ſquare of EA is double of the ſquare of AC. again,
becauſe GF is equal to FE, the ſquare of GF ſhall be equal to
the ſquare of FE; and therefor the ſquares of GF, FE are double of
the ſquare of EF. but the ſquare of EG is equal [i] to the ſquares of GF,
FE; therefor the ſquare of EG is double of the ſquare of EF. and
EF is equal to CD, wherefor the ſquare of EG is double of the
ſquare of CD. but it was demonſtrated that the ſquare of EA is
double



double of the ſquare of AC; therefor the ſquares of AE, EG are double of the ſquares of AC, CD. and the ſquare of AG is equal [i] to the ſquares of AE, EG; therefor the ſquare of AG is double of the ſquares of AC, CD. but the ſquares of AD, DG are equal [i] to the ſquare of AG; therefor the ſquares of AD, DG are double of the ſquares of AC, CD. but DG is equal to DB; therefor the ſquares of AD, DB are double of the ſquares of AC, CD. wherefor if a ſtraight line &c. Q. E. D.

i. 47. 1.

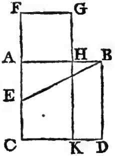
PROP. XI. PROB.

TO divide a given ſtraight line into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

Let AB be the given ſtraight line; it is required to divide it into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

Upon AB deſcribe [a] the ſquare ABDC, biſſect [b] AC in E, and join BE; produce CA to F, and make [c] EF equal to EB; and upon AF deſcribe [a] the ſquare FGHA, and produce GH to K. AB ſhall be divided in H ſo, that the rectangle AB, BH is equal to the ſquare of AH.

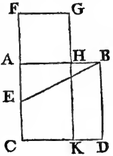
a. 46. 1.
b. 10. 1.
c. 3. 1.



Becauſe the ſtraight line AC is biſſected in E, and produced to the point F, the rectangle CF, FA, together with the ſquare of AE, ſhall be equal [d] to the ſquare of EF. but EF is equal to EB; therefor the rectangle CF, FA, together with the ſquare of AE is equal to the ſquare of EB. and the

d. 6. 2.

ſquares of BA, AE are equal [e] to the ſquare of EB, becauſe the angle e. 47. 1. EAB is a right angle; therefor the rectangle CF, FA, together with the ſquare of AE is equal to the ſquares of BA, AE. take away the ſquare of AE, which is common to both, and the remaining rectangle CF, FA ſhall be equal to the ſquare of AB. and the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the ſquare of AB; therefor FK is equal to AD. take away the common part AK, and the remainder FH ſhall be equal to the remainder HD. and HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the ſquare of AH. therefor the rectangle AB, BH is equal to the ſquare of AH. wherefor the ſtraight line AB is divided in H, ſo that the rectangle AB, BH is equal to the ſquare of AH. Which was to be done.



PROP. XII. THEOR.

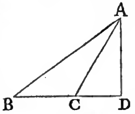
IN obtuſe angled triangles, if a perpendicular be drawn from any of the acute angles to the oppoſite ſide produced, the ſquare of the ſide ſubtending the obtuſe angle, is greater than the ſquares of the ſides containing the obtuſe angle, by twice the rectangle contained by the ſide upon which when produced the perpendicular falls, and the ſtraight line intercepted without the triangle between the perpendicular and the obtuſe angle.

Let ABC be an obtuſe angled triangle, having the obtuſe angle a. 12. 1. ACB, and from the point A let AD be drawn [a] perpendicular to BC produced. the ſquare of AB ſhall be greater than the ſquares of AC, CB by twice the rectangle BC, CD.

Becauſe the ſtraight line BD is divided into two parts in the point C, the ſquare of BD ſhall be equal [b] to the ſquares of BC, CD, and twice the rectangle BC, CD. to each of theſe equals add the ſquare of DA; and the ſquares of BD, DA ſhall be equal to the ſquares of BC, CD, DA, and twice the rectangle BC, CD. but the ſquare of BA is equal [c] to the ſquares of BD, DA, becauſe the angle at D is a right angle; and the ſquare of CA is equal [c] to the ſquares of CD, DA. therefor the ſquare of BA is equal to the ſquares of BC, CA, and twice the rectangle BC, CD; that is, the ſquare of BA is greater than the ſquares of BC, CA, by twice the rectangle BC, CD. therefor in obtuſe angled triangles &c. Q. E. D.

b. 4. 2.

c. 47. 1.



PROP. XIII. THEOR.

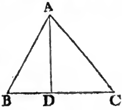
IN every triangle the ſquare of the ſide ſubtending any of the acute angles, is leſs than the ſquares of the ſides containing that angle, by twice the rectangle contained by any of theſe ſides, and the ſtraight line intercepted between the perpendicular let fall upon it, or upon it produced, from the angle oppoſite to it, and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC one of the ſides containing it let fall the perpendicular [a] AD from the oppoſite angle. the ſquare of AC oppoſite to the angle

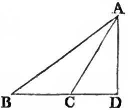
a. 12. 1.

angle B, ſhall be leſs than the ſquares of CB, BA by twice the rectangle CB, BD.

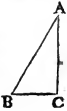
Firſt, Let AD fall within the triangle ABC; and becauſe the ſtraight line CB is divided into two parts in the point
b. 7. 2. D, the ſquares of CB, BD ſhall be equal [b] to twice the rectangle contained by CB, BD, and the ſquare of DC. to each of theſe equals add the ſquare of AD, and the ſquares of CB, BD, DA ſhall be equal to twice the rectangle CB, BD, and the ſquares of AD,
c. 47. 1. DC. but the ſquare of AB is equal [c] to the ſquares of BD, DA, becauſe the angle BDA is a right angle; and the ſquare of AC is equal to the ſquares of AD, DC. therefor the ſquares of CB, BA are equal to the ſquare of AC, and twice the rectangle CB, BD; that is, the ſquare of AC alone is leſs than the ſquares of CB, BA by twice the rectangle CB, BD.



But let AD fall without the triangle ABC. therefor becauſe the
d. 16. 1. angle at D is a right angle, the angle ACB ſhall be greater [d] than a right angle; and therefor the ſquare of AB
e. 12. 2. ſhall be equal [e] to the ſquares of AC, CB and twice the rectangle BC, CD. to theſe equals add the ſquare of BC, and the ſquares of AB, BC ſhall be equal to the ſquare of AC, and twice the ſquare of BC, and twice the rectangle BC, CD. but becauſe BD is divided in-
f. 3. 2. to two parts in C, the rectangle DB, BC is equal [f] to the rectangle BC, CD and the ſquare of BC. and the doubles of theſe are equal. therefor the ſquares of AB, BC are equal to the ſquare of AC, and twice the rectangle DB, BC. therefor the ſquare of AC alone, is leſs than the



the ſquares of AB, BC, by twice the rectangle DB, BC.



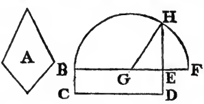
Laſtly, Let the ſide AC be perpendicular to BC; then is BC the ſtraight line between the perpendicular and the acute angle at B. and it is manifeſt that the ſquares of AB, BC are equal [c] to the ſquare of AC, and twice the ſquare of BC. therefor in every triangle &c. Q. E. D. c. 47. 1.

PROP. XIV. PROB.

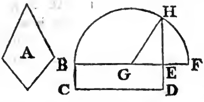
TO deſcribe a ſquare that ſhall be equal to a given rectilineal figure.

Let A be the given rectilineal figure; it is required to deſcribe a ſquare that ſhall be equal to A.

Deſcribe [a] the rectangular parallelogram BCDE equal to the recti- a. 45. 1.
lineal figure A. If then the ſides of it BE, ED are equal to one another, it is a ſquare, and what was required is now done. but if they are not equal, produce one of them BE to F, and make EF equal to ED, and biſſect BF in G; and from the centre G, at the diſtance GB or GF deſcribe the ſemicircle BHF, and produce DE to H, and join GH. therefor becauſe the ſtraight line BF is divided into two equal parts in the point G, and into two unequal at E, the rectangle BE, EF, together with the ſquare of EG, ſhall be equal [b] to the ſquare of GF. but GF is equal to GH; therefor the b. 5. 2.
rectangle BE, EF, together with the ſquare of EG, is equal to the ſquare



ſquare of GH. but the ſquares of HE, EG are equal [c] to the ſquare of GH. therefor the rectangle BE, EF together with the ſquare of EG is equal to the ſquares of HE, EG. take away the ſquare of EG, which is common to both; and the remaining rectangle BE, EF ſhall be equal to the ſquare of EH. but the rectangle contained by BE, EF is the parallelogram BD, becauſe EF is equal to ED; therefor BD is equal to the ſquare of EH. but BD is equal to the rectilineal figure A; therefor the rectilineal figure A ſhall be equal to the ſquare of EH. wherefor to the given rectilineal figure A there has been made a ſquare equal, viz. the ſquare deſcribed upon EH. Which was to be done.



[c] c. 47. 1.

THE ELEMENTS OF EUCLID.

BOOK III.

DEFINITIONS.

I.

EQUAL circles are thoſe whoſe diameters are equal, or from whoſe centres the ſtraight lines to the circumferences are equal.

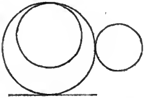
'This is not a Definition but a Theorem, the truth of which is 'evident; for if the circles be applied to one another, ſo that their 'centres coincide, the circles ſhall likewiſe coincide, ſince the ſtraight 'lines from their centres are equal.'

II.

A ſtraight line is ſaid to touch a circle, when it meets the circle and being produced does not cut it.

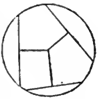
III.

Circles are ſaid to touch one another, which meet but do not cut one another.



IV.

Straight lines are ſaid to be equally diſtant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



V.

But the ſtraight line on which the greater perpendicular falls, is ſaid to be further from the centre.

VI.

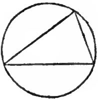
A ſegment of a circle is the figure contained by a ſtraight line and the circumference it cuts off.

VII.

" The angle of a ſegment is that which is contained by the ſtraight " line and the circumference."

VIII.

An angle in a ſegment is the angle contained by two ſtraight lines drawn from any point in the circumference of the ſegment, to the extremities of the ſtraight line which is the baſe of the ſegment.

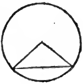


IX.

And an angle is ſaid to inſiſt or ſtand upon the circumference which the ſtraight lines that contain the angle intercept between them.

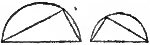
X.

The ſector of a circle is the figure contained by two ſtraight lines drawn from the centre, and the circumference between them.



XI.

Similar ſegments of a circle, are thoſe in which the angles are equal, or which contain equal angles.



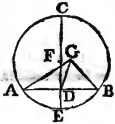
PROP. I. PROB.

TO find the centre of a given circle.

Let ABC be the given circle; it is required to find its centre.

Draw within it any ſtraight line AB, and biſſect [a] it in D; from a. 10. 1.
the point D draw [b] DC at right angles to AB, and produce it to E, b. 11. 1.
and biſſect CE in F. the point F ſhall be the centre of the circle ABC.

For if it be not, let, if poſſible, G be the centre, and join GA, GD, GB. therefor becauſe DA is equal to DB, and DG common to the two triangles ADG, BDG, the two ſides AD, DG are equal to the two BD, DG, each to each; and the baſe GA is equal to the baſe GB, becauſe they are from the centre G. therefor the angle ADG is equal [c] to the angle GDB. c. 8. 1.
but when a ſtraight line ſtanding upon another ſtraight line, makes the adjacent angles equal to one another, each of the angles is a right angle[d]. d. 10. Def. 1.



therefor the angle GDB is a right angle. but FDB is likewiſe a right angle; wherefor the angle FDB is equal to the angle GDB, the greater to the leſſer, which is impoſſible. therefor G is not the centre of the circle ABC. in the ſame manner it ſhall be ſhewn, that no other point but F is the centre; that is, F is the centre of the circle ABC. Which was to be found.

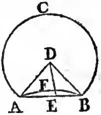
Cor. From this it is manifeſt, that if in a circle a ſtraight line biſſect another at right angles, the centre of the circle ſhall be in the line which biſſects the other.

PROP. II. THEOR.

IF any two points be taken in the circumference of a circle, the ſtraight line which joins them ſhall fall within the circle.

Let ABC be a circle, and A, B any two points in the circumference; the ſtraight line drawn from A to B ſhall fall within the circle.

a. 1. 3. For if it do not, let it fall, if poſſible without, as AEB; find [a] D
the centre of the circle ABC, and join AD, DB, and produce DF,
any ſtraight line meeting the circumference AB, to
E. therefor becauſe DA is equal to DB, the angle
b. 5. 1. DAB ſhall be equal [b] to the angle DBA; and becauſe
AE a ſide of the triangle DAE is produ-
c. 16. 1. ced to B, the angle DEB ſhall be greater [c] than the
angle DAE; but DAE is equal to the angle DBE,
therefor the angle DEB is greater than the angle
d. 19. 1. DBE. but to the greater angle the greater ſide is oppoſite[d]; DB is
therefor greater than DE. but DB is equal to DF; wherefor DF is
greater than DE, the leſſer than the greater, which is impoſſible. therefor
the ſtraight line drawn from A to B does not fall without the
circle. in the ſame manner, it ſhall be demonſtrated that it does not
fall upon the circumference. it ſhall therefor fall within it. wherefor
if any two points &c. Q. E. D.



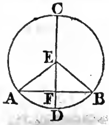
PROP. III.

PROP. III. THEOR.

IF a ſtraight line drawn thro' the centre of a circle, biſſect a ſtraight line in it which does not paſs thro' the centre, it ſhall cut it at right angles. and if it cuts it at right angles, it ſhall biſſect it.

Let ABC be a circle; and let CD a ſtraight line drawn thro' the centre biſſect any ſtraight line AB, which does not paſs thro' the centre, in the point F. it ſhall cut AB at right angles.

Take [a] E the centre of the circle, and join EA, EB. then becauſe a. 1. 3. AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two ſides in the one equal to two ſides in the other, and the baſe EA is equal to the baſe EB; therefor the angle AFE is equal [b] to the angle BFE. b. 8. 1. but when a ſtraight line ſtanding upon another makes the adjacent angles equal to one another, each of them is a right [c] angle. c. 10. Def. 1. therefor each of the angles AFE, BFE is a right angle; wherefor the ſtraight line CD drawn thro' the centre which biſſects another AB that does not paſs thro' the centre, cuts the ſame at right angles.



But let CD cut AB at right angles, CD ſhall alſo biſſect it; that is, AF ſhall be equal to FB.

The ſame conſtruction being made, becauſe EA, EB from the centre are equal to one another, the angle EAF ſhall be equal [d] to the angle d. 5. 1. EBF; and the right angle AFE is equal to the right angle BFE. therefor in the two triangles EAF, EBF there are two angles in one equal to two angles in the other, and the ſide EF which is oppoſite

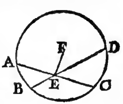
to one of the equal angles in each, is common to both; therefor the
c. 26. 1. other ſides ſhall be equal[c], AF therefor is equal to FB. wherefor if a ſtraight line &c. Q. E. D.

PROP. IV. THEOR.

IF in a circle two ſtraight lines cut one another which do not both paſs thro' the centre, they ſhall not biſſect each the other.

Let ABCD be a circle, and AC, BD two ſtraight lines in it which cut one another in the point E, and do not both paſs thro' the centre. AC, BD ſhall not biſſect one another.

For, if it is poſſible, let AE be equal to EC, and BE to ED. if one of the lines paſs thro' the centre, it is plain that it cannot be biſſected by the other which does not paſs thro' the centre. but if neither of them paſs thro'
a. 1. 3. the centre, take [a] F the centre of the circle, and join EF. and becauſe EF a ſtraight line thro' the centre, biſſects another AC which does not paſs thro' the centre, it ſhall cut it at
b. 3. 3. right [b] angles; wherefor FEA is a right angle. again, becauſe the ſtraight line FE biſſects the ſtraight line BD which does not paſs thro' the centre, it ſhall cut it at right [b] angles; wherefor FEB is a right angle. and FEA was ſhewn to be a right angle; therefor FEA is equal to the angle FEB, the leſs to the greater, which is impoſſible. therefor AC, BD do not biſſect one another. wherefor if in a circle &c. Q. E. D.

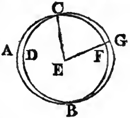


PROP. V. THEOR.

IF two circles cut one another, they ſhall not have the ſame centre.

Let the two circles ABC, CDG cut one another in the points B, C; they ſhall not have the ſame centre.

For, if it be poſſible, let E be their centre; join EC, and draw any ſtraight line EFG meeting them in F and G. and becauſe E is the centre of the circle ABC, CE ſhall be equal to EF. again, becauſe E is the centre of the circle CDG, CE is equal to EG. but CE was ſhewn to be equal to EF; therefor EF is equal to EG, the leſs to the greater, which is impoſſible. therefor E is not the centre of the circles ABC, CDG. wherefor if two circles &c. Q. E. D.

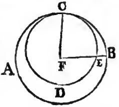


PROP. VI. THEOR.

IF two circles touch one another within, they ſhall not have the ſame centre.

Let the two circles ABC, CDE touch one another within in the point C. they ſhall not have the ſame centre.

For if they can, let it be F; join FC, and draw any ſtraight line FEB meeting them in E and B. and becauſe F is the centre of the circle ABC, CF is equal to FB. alſo becauſe F is the centre of the circle CDE, CF is equal

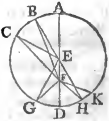


equal to FE. and CF was ſhewn equal to FB; therefor FE is equal to FB, the leſs to the greater, which is impoſſible. wherefor F is not the centre of the circles ABC, CDE. therefor if two circles &c. Q. E. D.

PROP. VII. THEOR.

IF any point be taken in the diameter of a circle, which is not the centre, of all the ſtraight lines which can be drawn from it to the circumference, the greateſt is that in which the centre is, and the other part of that diameter is the leaſt; and of any others, that which is nearer to the line which paſſes thro' the centre is ever greater than one more remote. and from the ſame point there can be drawn only two that are equal to one another, one upon each ſide of the ſhorteſt line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre. let the centre be E; of all the ſtraight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA is the greateſt, and FD the other part of the diameter AD is the leaſt; and of the others, FB is greater than FC, and FC than FG.



Join BE, CE, GE; and becauſe two ſides of a triangle are greater [a] than the third, BE, EF ſhall be greater than BF; but AE is equal to EB, therefor AE, EF, that is AF, is greater than BF. again, becauſe BE is equal to CE, and FE common to the triangles BEF, CEF; the two ſides BE, EF are equal to the two CE, EF; but

[a] 20. 1.

but the angle BEF is greater than the angle CEF, therefor the baſe
BF is greater [b] than the baſe FC. by the ſame reaſon, CF is greater than b. 24. 1.
GF. again, becauſe GF, FE are greater [a] than EG, and EG is equal a. 20. 1.
to ED; GF, FE ſhall be greater than ED. take away the common
part FE, and the remainder GF ſhall be greater than the remainder
FD. therefor FA is the greateſt, and FD the leaſt of all the ſtraight
lines from F to the circumference; and BF is greater than CF, and
CF than GF.

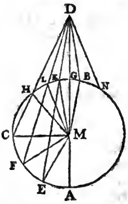
Alſo there can be drawn only two equal ſtraight lines from the
point F to the circumference, one upon each ſide of the ſhorteſt line
FD. at the point E in the ſtraight line EF, make [c] the angle FEH c. 23. 1.
equal to the angle GEF, and join FH. then becauſe GE is equal to
EH, and EF common to the two triangles GEF, HEF; the two ſides
GE, EF are equal to the two HE, EF; and the angle GEF is equal
to the angle HEF, therefor the baſe FG is equal [d] to the baſe FH. d. 4. 1.
but beſides FH no other ſtraight line can be drawn from F to the circumference equal to FG. for if there can, let it be FK, and becauſe FK is equal to FG, and FG to FH, FK ſhall be equal to FH, that is, a line nearer to that which paſſes thro' the centre is equal to one which is more remote; which is impoſſible. therefor if any point be taken &c. Q. E. D.

PROP. VIII. THEOR.

IF any point be taken without a circle, and ſtraight lines be drawn from it to the circumference, whereof one paſſes thro' the centre; of thoſe which fall upon the concave circumference the greateſt is that which paſſes thro' the centre; and of the reſt, that which is nearer to the one thro' the centre is ever greater than the more remote.

but of thoſe which fall upon the convex circumference, the leaſt is that between the point without the circle, and the diameter; and of the reſt, that which is nearer to the leaſt is ever leſs than the more remote. and only two equal ſtraight lines can be drawn from the point unto the circumference, one upon each ſide of the leaſt.

Let ABC be a circle, and D any point without it, from which let the ſtraight lines DA, DE, DF, DC be drawn to the circumference, whereof DA paſſes thro' the centre. of thoſe which fall upon the concave part of the circumference AEFC, the greateſt ſhall be AD which paſſes thro' the centre; and the nearer to it is ever greater than the more remote, viz. DE than DF, and DF than DC. but of thoſe which fall upon the convex circumference HLKG, the leaſt is DG between the point D and the diameter AG; and the nearer to it is ever leſſer than the more remote, viz. DK than DL, and DL than DH.



a. 1. 3. Find [a] M the centre of the circle ABC, and join ME, MF, MC, MK, ML, MH. and becauſe AM is equal to ME, add MD to each, and AD ſhall be equal to EM, MD; b. 20. 1. but EM, MD are greater [b] than ED, therefor alſo AD is greater than ED. again, becauſe ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD are equal to FM, MD; but the angle EMD is c. 24. 1. greater than the angle FMD, therefor the baſe ED is greater [c] than the baſe FD. in the ſame manner, FD ſhall be greater than CD. therefor DA is the greateſt; and DE greater than DF, and DF than

DC. and becauſe MK, KD are greater [b] than MD, and MK is equal to MG, the remainder KD ſhall be greater [d] than the remainder GD, that is, GD is leſs than KD. and becauſe MK, DK are drawn to the point K within the triangle MLD from M, D the extremities of its ſide MD; MK, KD ſhall be leſſer [e] than ML, LD, whereof MK is equal to ML, therefor the remainder DK is leſs than the remainder DL. in like manner, DL ſhall be leſſer than DH. therefor DG is the leaſt, and DK leſs than DL, and DL than DH. Alſo there can be drawn only two equal ſtraight lines from the point D to the circumference, one upon each ſide of the leaſt. at the point M in the ſtraight line MD, make the angle DMB equal to the angle DMK, and join DB. and becauſe MK is equal to MB, and MD common to the triangles KMD, BMD, the two ſides KM, MD are equal to the two BM, MD; and the angle KMD is equal to the angle BMD, therefor the baſe DK is equal [f] to the baſe DB. but beſides DB there can be no ſtraight line drawn from D to the circumference which ſhall be equal to DK. for if there can, let it be DN; and becauſe DK is equal to DN, and alſo to DB, DB ſhall be equal to DN, that is the nearer to the leaſt equal to the more remote, which is impoſſible. If therefor any point &c. Q. E. D.

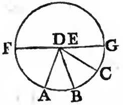
b. 20. 1.
d. 4. Ax.
e. 21. 1.
f. 4. 1.

PROP. IX. THEOR.

IF a point be taken within a circle, from which there fall more than two equal ſtraight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal ſtraight lines, viz. DA, DB, DC. the point D is the centre of the circle.

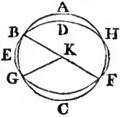
For if not, let E be the centre, join DE and produce it to the circumference in F, G; then FG is a diameter of the circle ABC. and becauſe in FG the diameter of the circle ABC there is taken the point D which is not the centre, DG ſhall be the greateſt line from it to the circumference, and DC ſhall be greater than DB, and DB than DA. but they are likewiſe equal, which is impoſſible. therefor E is not the centre of the circle ABC. in the ſame manner it ſhall be demonſtrated that no other point but D is the centre; D therefor is the centre. wherefor if a point be taken &c. Q. E. D.



PROP. X. THEOR.

ONE circle cannot cut another in more than two points.

If it be poſſible, let the circle ABC cut the circle DEF in more than two points, viz. in B, G, F; take the centre K of the circle ABC, and join KB, KG, KF. and becauſe within the circle DEF there is taken the point K from which to the circumference DEF fall more than two equal a. 9. 3. ſtraight lines KB, KG, KF, the point K ſhall [a] be the centre of the circle DEF. but K is alſo the centre of the circle ABC; therefor the ſame point is the centre of two circles that cut one b. 5. 3. another, which is impoſſible[b]. therefor one circle cannot cut another in more than two points. Q. E. D.

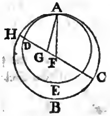


PROP. XI.

PROP. XI. THEOR.

IF two circles touch one another within, the ſtraight line which joins their centres being produced ſhall paſs through the point of contact.

Let the two circles ABC, ADE touch one another within in the point A, and let F be the centre of the circle ABC, and G the centre of the circle ADE. the ſtraight line which joins the centres F, G ſhall produced paſs thro' the point A.



For if not, let it fall otherwiſe, if poſſible, as FGDH, and join AF, AG. and becauſe AG, GF are greater [a] than FA, that is than FH, for FA is equal to FH, both being from the ſame centre; take away the common part FG, and the remainder AG ſhall be greater than the remainder GH. but AG is equal to GD, therefor GD is greater than GH, the leſſer than the greater, which is impoſſible. therefor the ſtraight line which joins the points F, G cannot fall otherwiſe than upon the point A, that is, it muſt paſs thro' it. therefor if two ſtraight lines &c. Q. E. D.

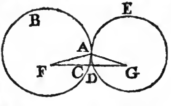
[a] 20. 1.

PROP. XII. THEOR.

IF two circles touch one another on the outſide, the ſtraight line which joins their centres ſhall paſs thro' the point of contact.

Let the two circles ABC, ADE touch one another on the outſide in the point A; and let F be the centre of the circle ABC, and G the centre of ADE. the ſtraight line which joins the points F, G ſhall paſs thro' the point of contact A.

For if not, let it paſs otherwiſe, if poſſible, as FCDG, and join FA, AG. and becauſe F is the centre of the circle ABC, AF ſhall be equal to FC. alſo becauſe G is the centre of the circle ADE, AG ſhall be equal to GD. therefor FA, AG are equal to FC, DG; wherefor the whole FG is

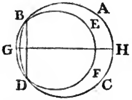


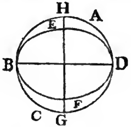
a. 20. 1. greater than FA, AG. but it is alſo leſs[a]; which is impoſſible. therefor the ſtraight line which joins the points F, G ſhall not paſs otherwiſe than thro' the point of contact A, that is, it muſt paſs thro' it. therefor if two circles &c. Q. E. D.

PROP. XIII. THEOR.

ONE circle cannot touch another in more points than one, whether it touches it on the inſide or outſide.

For, if it be poſſible, let the circle EBF touch the circle ABC in



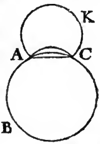


more points than one, and firſt on the inſide, in the points B, D; join
* 10, 11. 1. BD, and draw * GH biſſecting BD at right angles. therefor becauſe the points

points B, D are in the circumference of each of the circles, the ſtraight line BD ſhall fall within [a] each of them. and their centres ſhall be [b] in the ſtraight line GH which biſſects BD at right angles; therefor GH ſhall paſs thro' the point of contact[c]. but it does not paſs thro' it, becauſe the points B, D are without the ſtraight line GH, which is abſurd. therefor one circle cannot touch another on the inſide in more points than one.

a. 2. 3.
b. Cor. 1. 3.
c. 11. 3.

Nor can two circles touch one another on the outſide in more than one point. for, if it be poſſible, let the circle ACK touch the circle ABC in the points A, C, and join AC. therefor becauſe the two points A, C are in the circumference of the circle ACK, the ſtraight line AC which joins them ſhall fall within [a] the circle ACK. and the circle ACK is without the circle ABC, and therefor the ſtraight line AC is without this laſt circle; but becauſe the points A, C are in the circumference of the circle ABC, the ſtraight line AC muſt be within [a] the ſame circle, which is abſurd. therefor one circle cannot touch another on the outſide in more than one point. and it has been ſhewn that they cannot touch on the inſide in more points than one. therefor one circle &c. Q. E. D.

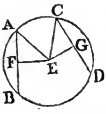


PROP. XIV. THEOR.

EQUAL ſtraight lines in a circle are equally diſtant from the centre; and thoſe which are equally diſtant from the centre, are equal to one another.

Let the ſtraight lines AB, CD in the circle ABDC be equal to one another; they ſhall be equally diſtant from the centre.

Take E the centre of the circle ABDC, and from it draw EF,
EG perpendicular to AB, CD. then becauſe the ſtraight line EF paſſing
thro' the centre cuts the ſtraight line AB, which does not paſs thro'
a. 3. 3. the centre, at right angles, it ſhall alſo biſſect [a] it. wherefor AF is equal
to FB, and AB ſhall be double of AF. by the ſame reaſon CD is
double of CG. and AB is equal to CD, therefor AF is equal to CG.
and becauſe AE is equal to EC, the ſquare of AE ſhall be equal to
b. 47. 1. the ſquare of EC. but the ſquares of AF, FE are equal [b] to the
ſquare of AE, becauſe the angle AFE is a right angle; and for the
like reaſon the ſquares of EG, GC are equal to the ſquare of EC.
therefor the ſquares of AF, FE are equal to the
ſquares of CG, GE, of which the ſquare of AF
is equal to the ſquare of CG, becauſe AF is equal
to CG; therefor the remaining ſquare of FE
is equal to the remaining ſquare of EG, and the
ſtraight line FE is therefor equal to EG. but
ſtraight lines in a circle are ſaid to be equally
diſtant from the centre, when the perpendiculars drawn to them from
c. 4. Def. 3. the centre are equal[c]. therefor AB, CD are equally diſtant from the
centre.



Next, if the ſtraight lines AB, CD be equally diſtant from the centre, that is, if FE be equal to EG; AB ſhall be equal to CD. for, the ſame conſtruction being made, it ſhall, as before, be demonſtrated that AB is double of AF and CD double of CG, and that the ſquares of EF, FA are equal to the ſquares of EG, GC; of which the ſquare of FE is equal to the ſquare of EG, becauſe FE is equal to EG; therefor the remaining ſquare of AF is equal to the remaining ſquare of CG; and the ſtraight line AF is therefor equal to CG. and AB is double of AF, and CD double of CG; wherefor AB is equal to CD. Therefor equal ſtraight lines &c. Q. E. D.

PROP. XV. THEOR.

THE diameter is the greateſt ſtraight line in a circle; and of any others, that which is nearer to the centre is ever greater than one more remote; and the greater is nearer to the centre than the leſſer.

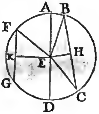
Let ABCD be a circle, whoſe diameter is AD, and centre E; and let BC be nearer to the centre than FG. AD ſhall be greater than BC any ſtraight line which is not a diameter, and BC ſhall be greater than FG.

From the centre draw EH, EK perpendicular to BC, FG, and join EB, EC, EF; and becauſe AE is equal to EB, and ED to EC, AD ſhall be equal to EB, EC. but EB, EC, are greater[a] than BC, wherefor alſo AD ſhall be greater than BC.

a. 20. 1.

And becauſe BC is nearer to the centre than FG, EH ſhall be leſſer[b] than EK. but, as was demonſtrated in the preceding, BC is double of BH, and FG double of FK, and the ſquares of EH, HB equal to the ſquares of EK, KF, of which the ſquare of EH is leſſer than the ſquare of EK, becauſe EH is leſſer than EK; therefor the ſquare of BH ſhall be greater than the ſquare of FK, and the ſtraight line BH greater than FK; and therefor BC is greater than FG.

b. 5. Def. 3.



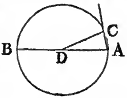
Next, let BC be greater than FG, BC ſhall be nearer to the centre than FG, that is, the ſame conſtruction being made, EH ſhall be leſs than EK. for becauſe BC is greater than FG, likewiſe BH ſhall be greater than FK. and the ſquares of BH, HE are equal to the ſquares of FK, KE, of which the ſquare of BH is greater than the ſquare of FK, becauſe BH is greater than FK; therefor the ſquare of EH is leſs than the ſquare of EK, and the ſtraight line EH ſhall be leſs than EK. wherefor the diameter &c. Q. E. D.

PROP. XVI. THEOR.

THE ſtraight line which is drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no ſtraight line can be drawn from the extremity, between that ſtraight line and the circumference which does not cut the circle, or, which is the ſame thing, no ſtraight line can make ſo great an acute angle with the diameter at its extremity, or ſo ſmall an angle with the ſtraight line which is at right angles to it, but that it ſhall cut the circle.

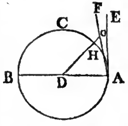
Let ABC be a circle whoſe centre is D, and diameter AB; the ſtraight line drawn at right angles to AB from its extremity A, ſhall fall without the circle.

For if it does not, let it fall, if poſſible, within the circle as AC,
and draw DC to the point C where it meets
the circumference. and becauſe DA is equal to
a. 5. 1. DC, the angle DAC ſhall be equal [a] to the
angle ACD; but DAC is a right angle, there-
for ACD is a right angle, and the angles DAC,
ACD ſhall be equal to two right angles; which
b. 17. 1. is impoſſible[b]. therefor the ſtraight line drawn from A at right angles
to BA does not fall within the circle. in the ſame manner it ſhall be
demonſtrated



demonſtrated that it does not fall upon the circumference; therefor it muſt fall without the circle, as AE.

And between the ſtraight line AE and the circumference no ſtraight line can be drawn from the point A which does not cut the circle. for, if poſſible, let FA be between them, and from the point D draw [c] DG c. 12. 1. perpendicular to FA, and let it meet the circumference in H. and becauſe AGD is a right angle, and DAG is leſs [b] than a right angle, b. 17. 1. DA ſhall be greater [d] than DG. but DA is e- d. 19. 1. qual to DH; therefor DH is greater than DG, the leſs than the greater, which is impoſſible. therefor no ſtraight line can be drawn from the point A between AE and the circumference, which does not cut the circle. or, which amounts to the ſame thing, however great an acute angle a ſtraight line makes with the diameter at the point A, or however ſmall an angle it makes with AE, the circumference ſhall paſs between that ſtraight line and the perpendicular AE. 'And this is all that is to be underſtood, when in the 'Greek text and tranſlations from it, the angle of the ſemicircle is ſaid 'to be greater than any acute rectilineal angle, and the remaining angle 'leſſer than any rectilineal angle.'



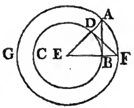
Cor. From this it is manifeſt that the ſtraight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, becauſe if it did meet the circle in two, it would fall within it[e]. 'Alſo it is evi- e. 2. 3. 'dent that there can be but one ſtraight line which touches the circle 'in the ſame point.'

PROP. XVII. PROB.

To draw a ſtraight line from a given point, which ſhall touch a given circle.

Let A be the given point, and BCD the given circle; it is required to draw a ſtraight line from A which ſhall touch the circle.

a. 1. 3. Find [a] the centre E of the circle, and join AE; and from the centre E, at the diſtance EA deſcribe the circle AFG; from the point D b. 11. 1. draw [b] DF at right angles to EA, and join EBF, AB. AB ſhall touch the circle BCD.



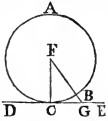
Becauſe E is the centre of the circles BCD, AFG, EA ſhall be equal to EF, and ED to EB; therefor the two ſides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefor the baſe DF is equal to the baſe AB, and the triangle FED to the triangle AEB, and the c. 4. 1. other angles to the other angles[c]. therefor the angle EBA is equal to the angle EDF. but EDF is a right angle, wherefor EBA is a right angle. and EB is from the centre; but a ſtraight line drawn from the d. Cor. 16. 3. extremity of a diameter, at right angles to it, touches the circle[d]. therefor AB touches the circle; and it is drawn from the given point A. Which was to be done.

PROP. XVIII.

PROP. XVIII. THEOR.

IF a ſtraight line touches a circle, the ſtraight line drawn from the centre to the point of contact, ſhall be perpendicular to the line which touches the circle.

Let the ſtraight line DE touch the circle ABC in the point C, find the centre F, and draw the ſtraight line FC; FC ſhall be perpendicular to DE.



For if it be not, from the point F draw FBG perpendicular to DE; and becauſe FGC is a right angle, GCF ſhall [b] be an acute angle; and to the greater angle the greateſt [c] ſide is oppoſite. therefor FC is greater than FG; but FC is equal to FB; therefor FB is greater than FG, the leſs than the greater, which is impoſſible. wherefor FG is not perpendicular to DE. in the ſame manner it ſhall be ſhewn, that no other is perpendicular to it beſides FC, that is, FC is perpendicular to DE. therefor if a ſtraight line &c. Q. E. D.

b. 17. 1.
c. 19. 1.

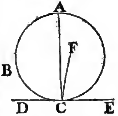
PROP. XIX. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn at right angles to the touching line, the centre of the circle ſhall be in that line.

Let the ſtraight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the centre of the circle ſhall be in CA.

For

Book III. For if not, let F be the centre, if poſſible, and join CF. Becauſe DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC a. 18. 3. ſhall be perpendicular [a] to DE; therefor FCE is a right angle. but ACE is alſo a right angle; therefor the angle FCE is equal to the angle ACE, the leſs to the greater, which is impoſſible. wherefor F is not the centre of the circle ABC. in the ſame manner it ſhall be ſhewn that no other point which is not in CA, is the centre; that is, the centre is in CA. therefor if a ſtraight line &c. Q. E. D.

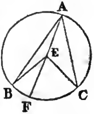


PROP. XX. THEOR.

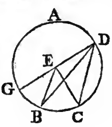
THE angle at the centre of a circle is double of the angle at the circumference upon the ſame baſe, that is, upon the ſame part of the circumference.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the ſame circumference BC for their baſe; the angle BEC ſhall be double of the angle BAC.

Firſt, Let E the centre of the circle be within the angle BAC, and join AE, and produce it to F. Becauſe EA is equal a. 5. 1. to EB, the angle EAB ſhall be equal [a] to the angle EBA; therefor the angles EAB, EBA ſhall be double of the angle EAB; but the angle BEF is b. 32. 1. equal [b] to the angles EAB, EBA; therefor alſo the angle BEF is double of the angle EAB. by the ſame reaſon, the angle FEC is double of the angle EAC. therefor the whole angle BEC is double of the whole angle BAC.



Again, Let BDC be inflected to the circumference, ſo that E the centre of the circle be without the angle BDC, and join DE and produce it to G. It ſhall be demonſtrated, as in the firſt caſe, that the angle GEC is double of the angle GDC, and that GEB a part of the firſt is double of GDB a part of the other; therefor the remaining angle BEC is double of the remaining angle BDC. therefor the angle at the centre &c. Q. E. D.

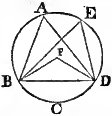


PROP. XXI. THEOR.

THE angles in the ſame ſegment of a circle are equal to one another.

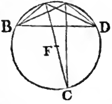
Let ABCD be a circle, and BAD, BED angles in the ſame ſegment BAED; the angles BAD, BED ſhall be equal to one another.

Take F the centre of the circle ABCD. and, firſt, let the ſegment BAED be greater than a ſemicircle, and join BF, FD. and becauſe the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the ſame part of the circumference, viz. BCD for their baſe, the angle BFD ſhall be double [a] of the angle BAD. by the ſame reaſon, the angle BFD is double of the angle BED. therefor the angle BAD is equal to the angle BED.



a. 20. 3.

But if the ſegment BAED be not greater than a ſemicircle, let BAD, BED be angles in it; theſe ſhall be equal to one another. draw AF to the centre, and produce it to C, and join CE. therefor the ſegment BADC is greater than a ſemicircle; and the angles in it BAC, BEC ſhall be equal, by the firſt caſe. by the ſame reaſon, the angles CAD, CED are equal. therefor the whole angle BAD is equal to the whole angle BED. wherefor the angles in the ſame ſegment &c. Q. E. D.

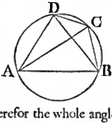


PROP. XXII. THEOR.

THE oppoſite angles of any quadrilateral figure deſcribed in a circle, are equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its oppoſite angles ſhall be equal to two right angles.

Join AC, BD; and becauſe the three angles of every triangle are a. 32. 1. equal [a] to two right angles, the three angles of the triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles. but the b. 21. 3. angle CAB is equal [b] to the angle CDB, becauſe they are in the ſame ſegment BADC; and the angle ACB is equal to the angle ADB, becauſe they are in the ſame ſegment ADCB. therefor the whole angle ADC is equal to the angles CAB, ACB. to each of theſe equals add the angle ABC, and the angles ABC, CAB, BCA ſhall be equal to the angles ABC, ADC. but ABC, CAB, BCA are equal to two right angles; therefor alſo the angles ABC, ADC are equal to two right angles.

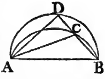


angles. in the ſame manner the angles BAD, DCB ſhall be ſhewn to be equal to two right angles. therefor the oppoſite angles &c. Q. E. D.

PROP. XXIII. THEOR.

UPON the ſame ſtraight line, and upon the ſame ſide of it, there cannot be two ſimilar ſegments of circles, which do not coincide with one another.

If it be poſſible, let the two ſimilar ſegments of circles, viz. ACB, ADB be upon the ſame ſide of the ſame ſtraight line AB, and not coincide with one another. therefor becauſe the circle ACB cuts the circle ADB in the two points A, B, they ſhall not cut one another in any other point[a]. one of the ſegments muſt therefor fall within the other; let ACB fall within ADB, and draw the ſtraight line BCD, and join CA, DA. and becauſe the ſegment ACB is ſimilar to the ſegment ADB, and that ſimilar ſegments of circles contain [b] equal angles; the angle ACB ſhall be equal to the angle ADB, the exterior to the interior, which is impoſſible[c]. therefor there cannot be two ſimilar ſegments of a circle upon the ſame ſide of the ſame line, which do not coincide. Q. E. D.



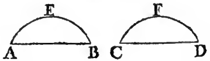
a. 10. 3.
b. 11. Def. 3.
c. 16. 1.

PROP. XXIV. THEOR.

SIMILAR ſegments of circles upon equal ſtraight lines, are equal to one another.

Let AEB, CFD be ſimilar ſegments of circles upon the equal ſtraight lines AB, CD; the ſegment AEB ſhall be equal to the ſegment CFD.

For if the ſegment AEB be applied to the ſegment CFD, ſo as the point A be in C, and the ſtraight line AB upon CD, the point B



ſhall coincide with the point D, becauſe AB is equal to CD. therefor
a. 23. 3. the ſtraight line AB coinciding with CD, the ſegment AEB muſt [a] coincide with the ſegment CFD, and therefor is equal to it. wherefor ſimilar ſegments &c. Q. E. D.

PROP. XXV. PROB.

A Segment of a circle being given, to deſcribe the circle of which it is the ſegment.

Let ABC be the given ſegment of a circle; it is required to deſcribe the circle of which it is the ſegment.

a. 10. 1. b. 11. 1. Biſſect [a] AC in D, and from the point D draw [b] DB at right angles to AC, and join AB. Firſt, let the angles ABD, BAD be equal to

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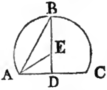

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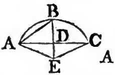

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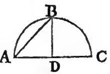


c. 6. 1. one another, and the ſtraight line BD ſhall be equal [c] to DA, and therefor to DC. and becauſe the three ſtraight lines DA, DB, DC
b. 9. 3. are all equal, D ſhall be the centre of the circle[d]. from the centre D, at

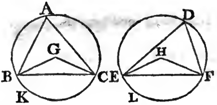
at the diſtance of any of the three DA, DB, DC deſcribe a circle; this ſhall paſs thro' the other points, and the circle of which ABC is a ſegment ſhall be deſcribed. and becauſe the centre D is in AC, the ſegment ABC ſhall be a ſemicircle. but if the angles ABD, BAD are not equal to one another, at the point A in the ſtraight line AB make [e] the angle BAE equal to the angle ABD, and produce BD to E, and e. 23. 1. join EC. and becauſe the angle ABE is equal to the angle BAE, the ſtraight line BE ſhall be equal [c] to EA. and becauſe AD is equal c. 6. 1. to DC, and DE common to the triangles ADE, CDE, the two ſides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefor the baſe AE is equal [f] to the baſe EC. but AE was ſhewn f. 4. 1. to be equal to EB, wherefor alſo BE is equal to EC; and the three ſtraight lines AE, EB, EC are therefor equal to one another; wherefor [d] E is the centre of the circle. from the centre E at the diſtance of d. 9. 3. any of the three AE, EB, EC deſcribe a circle, this ſhall paſs thro' the other points, and the circle of which ABC is a ſegment ſhall be deſcribed. and it is evident that if the angle ABD be greater than the angle BAD, the centre E ſhall fall without the ſegment ABC, which therefor ſhall be leſs than a ſemicircle. but if the angle ABD be leſs than BAD, the centre E ſhall be within the ſegment ABC which ſhall therefor be greater than a ſemicircle. wherefor a ſegment of a circle being given, the circle is deſcribed of which it is a ſegment. Which was to be done.

PROP. XXVI. THEOR.

IN equal circles, equal angles ſtand upon equal circumferences, whether they be at the centres, or circumferences.

Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences. the circumference BKC ſhall be equal to the circumference ELF.

Join BC, EF; and becauſe the circles ABC, DEF are equal, the ſtraight lines from their centres ſhall be equal; therefor the two ſides BG, GC, are equal to the two EH, HF; and the angle at G is equal



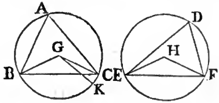
a. 4. 1. to the angle at H; therefor the baſe BC is equal [a] to the baſe EF. and becauſe the angle at A is equal to the angle at D, the ſegment b. 11. Def. 3. BAC is ſimilar [b] to the ſegment EDF; and they are upon equal ſtraight lines BC, EF; but ſimilar ſegments of circles upon equal ſtraight lines c. 24. 3. are equal [c] to one another; therefor the ſegment BAC is equal to the ſegment EDF. but the whole circle ABC is equal to the whole DEF, therefor the remaining ſegment BKC is equal to the remaining ſegment ELF, and the circumference BKC to the circumference ELF. Wherefor in equal circles &c. Q. E. D.

PROP. XXVII. THEOR.

IN equal circles, the angles which ſtand upon equal circumferences, are equal to one another, whether they be at the centres, or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF ſtand upon the equal circumferences BC, EF. the angle BGC ſhall be equal to the angle EHF, and the angle BAC to the angle EDF.

If the angle BGC be equal to the angle EHF, it is manifeſt[a] that a. 20. 3. likewiſe the angle BAC ſhall be equal to EDF. but if not, one of



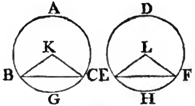
them is the greater. let BGC be the greater, and at the point G, in the ſtraight line BG, make[b] the angle BGK equal to the angle EHF; b. 23. 1. but equal angles ſtand upon equal circumferences[c], when they are at c. 26. 3. the centre; therefor the circumference BK is equal to the circumference EF. but EF is equal to BC, therefor alſo BK is equal to BC, the leſs to the greater, which is impoſſible. therefor the angle BGC is not unequal to the angle EHF; that is, it is equal to it. and the angle at A is half of the angle BGC, and the angle at D half of the angle EHF. therefor the angle at A is equal to the angle at D. wherefor in equal circles &c. Q. E. D.

PROP. XXVIII. THEOR.

IN equal circles, equal ſtraight lines cut off equal circumferences, the greater equal to the greater, and the leſſer to the leſſer.

Let ABC, DEF be equal circles, and BC, EF equal ſtraight lines in them, which cut off the two greater circumferences BAC, EDF, and the two leſſer BGC, EHF. the greater BAC ſhall be equal to the greater EDF, and the leſſer BGC to the leſſer EHF.

a. 1. 3. Take [a] K, L the centres of the circles, and join BK, KC, EL, LF. and becauſe the circles are equal, the ſtraight lines from their centres



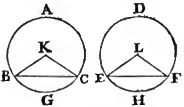
ſhall be equal, therefor BK, KC, are equal to EL, LF; and the baſe
b. 8. 1. BC is equal to the baſe EF; therefor the angle BKC is equal [b] to the
c. 26. 3. angle ELF. but equal angles ſtand upon equal [c] circumferences, when they are at the centres; therefor the circumference BGC is equal to the circumference EHF. but the whole circle ABC is equal to the whole EDF; the remaining part therefor of the circumference, viz. BAC is equal to the remaining part EDF. therefor in equal circles &c. Q. E. D.

PROP. XXIX. THEOR.

IN equal circles equal circumferences are ſubtended by equal ſtraight lines.

Let ABC, DEF be equal circles, and let equal circumferences BGC, EHF be taken in them; and join BC, EF. the ſtraight line BC ſhall be equal to the ſtraight line EF.

Take [a] K, L the centres of the circles, and join BK, KC, EL, LF. and becauſe the circumference BGC is equal to the circumference a. 1. 3.



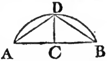
EHF, the angle BKC ſhall be equal [b] to the angle ELF. and be- b. 27. 3. cauſe the circles ABC, DEF are equal, the ſtraight lines from their centres ſhall be equal; therefor BK, KC are equal to EL, LF, and they contain equal angles. therefor the baſe BC is equal [c] to the baſe c. 4. 1. EF. therefor in equal circles &c. Q. E. D.

PROP. XXX. PROB.

TO biſſect a given circumference, *that is to divide it into two equal parts.*

Let ADB be the given circumference; it is required to biſſect ADB.

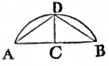
Join AB, and biſſect [a] it in C; from the point C draw CD at right a. 10. 1. angles to AB, and join AD, DB. the circumference ADB ſhall be biſſected in the point D.



Becauſe AC is equal to CB, and CD common to the triangles ACD, BCD, the two ſides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, becauſe each of them is a right angle; therefor the baſe AD is equal [b] to the baſe BD. but equal ſtraight lines cut off equal [c] b. 4. 1. circum- c. 28. 3.

circumferences, the greater equal to the greater, and the leſſer to the leſſer, and AD, DB are each of them leſſer than a ſemicircle; becauſe DC paſſes thro' the centre[d]. wherefor the circumference AD is equal to the circumference DB. therefor the given circumference is biſſected in D. Which was to be done.

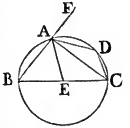
d. Cor. 1. 3.



PROP. XXXI. THEOR.

IN a circle, the angle in a ſemicircle is a right angle; but the angle in a ſegment greater than a ſemicircle is leſs than a right angle; and the angle in a ſegment leſſer than a ſemicircle is greater than a right angle.

Let ABCD be a circle, whoſe diameter is BC, and centre E; and draw CA dividing the circle into the ſegments ABC, ADC, and join BA, AD, DC. the angle in the ſemicircle BAC ſhall be a right angle; and the angle in the ſegment ABC, which is greater than a ſemicircle, ſhall be leſs than a right angle; and the angle in the ſegment ADC which is leſſer than a ſemicircle ſhall be greater than a right angle.



Join AE, and produce BA to F; and becauſe BE is equal to EA, the angle EAB ſhall be equal [a] to EBA; alſo, becauſe AE is equal to EC, the angle EAC ſhall be equal to ECA; wherefor the whole angle BAC ſhall be equal to the two angles ABC, ACB. but FAC the exterior angle of the triangle ABC, is equal [b] to the two angles ABC, ACB; therefor the angle BAC is equal to the angle FAC,

a. 5. 1.

b. 32. 1.

FAC, and each of them is therefor a right [c] angle. wherefor the angle BAC in a ſemicircle is a right angle. c. 10. Def. 1.

And becauſe the two angles ABC, BAC of the triangle ABC are together leſs [d] than two right angles, and that BAC is a right angle, ABC ſhall be leſs than a right angle; and therefor the angle in a ſegment ABC greater than a ſemicircle is leſs than a right angle. d. 17. 1.

And becauſe ABCD is a quadrilateral figure in a circle, any two of its oppoſite angles are equal [e] to two right angles; therefor the angles ABC, ADC are equal to two right angles; and ABC is leſſer than a right angle, wherefor the other ADC is greater than a right angle. e. 22. 3.

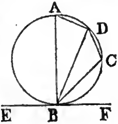
Beſides, it is manifeſt, that the circumference of the greater ſegment ABC falls without the right angle CAB, but the circumference of the leſſer ſegment ADC falls within the right angle CAF. ‘And this is all ‘that is meant, when in the Greek text, and the tranſlations from it, ‘the angle of the greater ſegment is ſaid to be greater, and the angle ‘of the leſſer ſegment is ſaid to be leſſer than a right angle.’

Cor. From this it is manifeſt, that if one angle of a triangle be equal to the other two, it is a right angle, becauſe the angle adjacent to it is equal to the ſame two; and when the adjacent angles are equal, they are right angles.

PROP. XXXII. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn cutting the circle, the angles which this makes with the line which touches the circle, ſhall be equal to the angles which are in the alternate ſegments of the circle.

Let the ſtraight line EF touch the circle ABCD in B, and from the point B let the ſtraight line BD be drawn cutting the circle. the angles which BD makes with the touching line EF ſhall be equal to the angles in the alternate ſegments of the circle; that is, the angle FBD ſhall be equal to the angle which is in the ſegment DAB, and the angle DBE to the angle in the ſegment BCD.

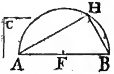


a. 11. 1. From the point B draw [a] BA at right angles to EF, and take any
point C in the circumference BD, and join AD, DC, CB; and becauſe
the ſtraight line EF touches the circle ABCD
in the point B, and BA is drawn at right angles
to the touching line from the point of contact
b. 19. 3. B, the centre of the circle ſhall be [b] in BA;
therefor the angle ADB in a ſemicircle is a
c. 31. 3. right [c] angle, and the other two angles BAD,
d. 32. 1. ABD ſhall be equal [d] to one right angle. but
ABF is likewiſe a right angle; therefor the angle
ABF is equal to the angles BAD, ABD. take from theſe equals the
common angle ABD, and the remaining angle DBF ſhall be equal to
the angle BAD which is in the alternate ſegment of the circle; and
becauſe ABCD is a quadrilateral figure in a circle, the oppoſite angles
e. 22. 3. BAD, BCD are equal [e] to two right angles; therefor the angles DBF,
f. 13. 1. DBE, which are likewiſe equal [f] to two right angles, ſhall be equal to
the angles BAD, BCD; and DBF has been proved equal to BAD;
therefor the remaining angle DBE is equal to the angle BCD in the
alternate ſegment of the circle. Wherefor if a ſtraight line &c.
Q. E. D.

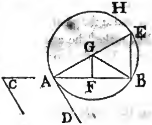
PROP. XXXIII. PROB.

TO deſcribe upon a given ſtraight line a ſegment of a circle, which ſhall contain an angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and the angle at C the given rectilineal angle; it is required to deſcribe upon the given ſtraight line AB a ſegment of a circle, which ſhall contain an angle equal to the angle C.

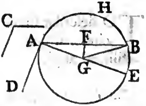


Firſt, Let the angle at C be a right angle, and biſſect [a] AB in F, and from the centre F, a. 10. 1.
at the diſtance FB deſcribe the ſemicircle AHB; therefor the angle AHB in a ſemicircle ſhall be [b] equal to the right angle at C. b. 31. 3.



But if the angle C be not a right angle, at the point A in the ſtraight line AB make [c] the angle BAD equal to the angle C, and c. 23. 1.
from the point A draw [d] AE at right angles to AD; biſſect [a] AB in F, and d. 11. 1.
from F draw [d] FG at right angles to AB, and join GB. and becauſe AF is equal to FB, and FG common to the triangles AFG, BFG, the two ſides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefor the baſe AG is equal [e] to the baſe GB; and the circle deſcribed from e. 4. 1.
the centre G, at the diſtance GA ſhall paſs thro' the point B; let this be the circle AHB. and becauſe from the point A the extremity of the diameter AE, AD is drawn at right angles to AE, AD ſhall

touch [f] the circle; and becauſe AB drawn from the point of contact f. Cor. 16. 3. A cuts the circle, the angle DAB ſhall be equal to the angle in the alternate ſegment AHB. but the angle DAB is equal to the angle C, therefor alſo the angle C ſhall be equal to the angle in the ſegment AHB. wherefor upon the given ſtraight line AB the ſegment AHB of a circle is deſcribed which contains an angle equal to the given angle at C. Which was to be done.

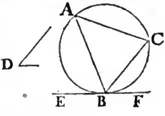


PROP. XXXIV. PROB.

TO cut off a ſegment from a given circle which ſhall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a ſegment from the circle ABC that ſhall contain an angle equal to the angle D.

a. 11. 1. Draw [a] the ſtraight line EF touching the circle ABC in the point B, and at the point B, in the ſtraight line b. 23. 1. BF, make [b] the angle FBC equal to the angle D. therefor becauſe the ſtraight line EF touches the circle ABC, and BC is drawn from the point of contact c. 32. 3. B, the angle FBC ſhall be equal [c] to the angle in the alternate ſegment BAC of the circle. but the angle FBC is equal to the angle D; therefor the angle in the ſegment BAC is equal to the angle D. wherefor the ſegment BAC is cut off from the given circle ABC containing an angle equal to the given angle D. Which was to be done.

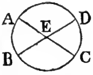


PROP. XXXV. THEOR.

IF two ſtraight lines cut one another within a circle, the rectangle contained by the ſegments of one of them, is equal to the rectangle contained by the ſegments of the other.

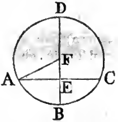
Let the two ſtraight lines AC, BD cut one another in the point E within the circle ABCD; the rectangle contained by AE, EC ſhall be equal to the rectangle contained by BE, ED.

If AC, BD paſs each of them thro' the centre, ſo that E is the centre; it is evident, that AE, EC, BE, ED being all equal, the rectangle AE, EC ſhall likewiſe be equal to the rectangle BE, ED.



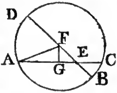
But let one of them BD paſs thro' the centre, and cut the other AC, which does not paſs thro' the centre, at right angles, in the point E. then if BD be biſſected in F, F ſhall be the centre of the circle ABCD; and to it from A draw AF. and becauſe BD which paſſes thro' the centre cuts the ſtraight line AC which does not paſs thro' the centre at right angles in E, AE, EC ſhall be equal [a] to one another. and becauſe the ſtraight line BD is cut into two equal parts in the point F, and into two unequal in the point E, the rectangle BE, ED together with the ſquare of EF, ſhall be equal [b] to the ſquare of FB; that is, to the ſquare of FA; but the ſquares of AE, EF are equal [c] to the ſquare of FA; therefor the rectangle BE, ED together with the ſquare of EF is equal to the ſquares of AE, EF. take away the common ſquare of EF, and the remaining rectangle BE,

a. 3. 3.
b. 5. 2.
c. 47. 1.

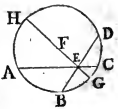


BE, ED ſhall be equal to the remaining ſquare of AE; that is, to the rectangle AE, EC.

Next, Let BD which paſſes thro' the centre, cut the other AC, which does not paſs thro' the centre, in E, but not at right angles. then, as before, if BD be biſſected in F, F ſhall be the centre of the
d. 12. 1. circle. Join AF, and from F draw [d] FG perpendicular to AC; there-
a. 3. 3. for AG is equal [a] to GC; wherefor the rectangle AE, EC together
b. 5. 2. with the ſquare of EG is equal [b] to the ſquare of AG. to each of theſe equals add the ſquare of GF, and the rectangle AE, EC together with the ſquares of EG, GF ſhall be equal to the ſquares of AG, GF. but the ſquares of EG,
c. 47. 1. GF are equal [c] to the ſquare of EF; and the ſquares of AG, GF are equal to the ſquare of AF. therefor the rectangle AE, EC together with the ſquare of EF is equal to the ſquare of AF; that is, to the ſquare of FB. but the ſquare of FB is equal [b] to the rectangle BE, ED together with the ſquare of EF; therefor the rectangle AE, EC together with the ſquare of EF, is equal to the rectangle BE, ED together with the ſquare of EF. take away the common ſquare of EF, and the remaining rectangle AE, EC ſhall be equal to the remaining rectangle BE, ED.



Laſtly, Let neither of the ſtraight lines AC, BD paſs thro' the centre. take the centre F, and thro' E the interſection of the ſtraight lines AC, DB draw the diameter GEFH. and becauſe the rectangle AE, EC is equal, as has been ſhewn, to the rectangle GE, EH; and by the ſame reaſon, the rectangle BE, ED is equal to the ſame rectangle GE, EH; the rectangle AE, EC ſhall be



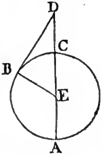
be equal to the rectangle BE, ED. therefor if two ſtraight lines &c. Q. E. D.

PROP. XXXVI. THEOR.

IF from any point without a circle two ſtraight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, ſhall be equal to the ſquare of the line which touches it.

Let D be any point without the circle ABC, and DCA, DB two ſtraight lines drawn from it, of which DCA cuts the circle, and DB touches the ſame. the rectangle AD, DC ſhall be equal to the ſquare of DB.

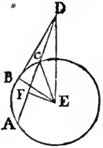
Either DCA paſſes thro' the centre, or it does not; firſt, let it paſs thro' the centre E, and join EB; the angle EBD ſhall be a right [a] angle. and becauſe the ſtraight line AC is biſſected in E, and produced to the point D, the rectangle AD, DC together with the ſquare of EC is equal [b] to the ſquare of ED. and CE is equal to EB, therefor the rectangle AD, DC together with the ſquare of EB is equal to the ſquare of ED. but the ſquare of ED is equal [c] to the ſquares of EB, BD, becauſe EBD is a right angle. therefor the rectangle AD, DC together with the ſquare of EB is equal to the ſquares of EB, BD. take away the common ſquare of EB, and the remaining rectangle AD, DC ſhall be equal to the ſquare of the tangent DB.



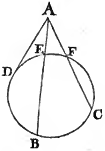
a. 18. 3.
b. 6. 2.
c. 47. 1.

But if DCA do not paſs thro' the centre of the circle ABC, take the

[d] the centre E, and draw EF perpendicular [e] to AC, and join EB,
d. 1. 3. EC, ED; then EFD is a right angle. and becauſe the ſtraight line EF
e. 12. 1. which paſſes thro' the centre, cuts the ſtraight line AC, which does
f. 3. 3. not paſs thro' the centre, at right angles, it ſhall likewiſe biſſect [f] it;
therefor AF is equal to FC. and becauſe the ſtraight line AC is biſſected in F, and produced to D, the rectangle AD, DC together with
b. 6. 2. the ſquare of FC ſhall be equal [b] to the ſquare of FD. to each of
theſe equals add the ſquare of FE, and the rectangle AD, DC together with the ſquares of CF, FE ſhall be equal to the ſquares of DF,
c. 47. 1. FE. but the ſquare of ED is equal [c] to the ſquares of DF, FE, becauſe EFD is a right angle; and the ſquare of EC is equal to the ſquares of CF, FE; therefor the rectangle AD, DC together with the ſquare of EC is equal to the ſquare of ED. and CE is equal to EB, therefor the rectangle AD, DC together with the ſquare of EB, is equal to the ſquare of ED. but the ſquares of EB, BD are equal to the ſquare[c] of ED, becauſe EBD is a right angle; therefor the rectangle AD, DC together with the ſquare of EB is equal to the ſquares of EB, BD. take away the common ſquare of EB, and the remaining rectangle AD, DC ſhall be equal to the ſquare of DB. therefor if from any point &c. Q. E. D.



Cor. If from any point without a circle there be drawn two ſtraight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, ſhall be equal to one another, viz. the rectangle BA, AE to the rectangle CA, AF. for each of them is equal to the ſquare of the ſtraight line AD which touches the circle.

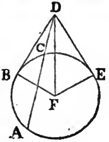


PROP. XXXVII. THEOR.

IF from a point without a circle there be drawn two ſtraight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the ſquare of the line which meets it, the line which meets ſhall touch the circle.

Let any point D be taken without the circle ABC, and from it let two ſtraight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD, DC be equal to the ſquare of DB; DB ſhall touch the circle.

Draw [a] the ſtraight line DE touching the circle ABC, find its centre
a. 17. 3. F, and join FE, FB, FD; then FED is a right [b] angle. and becauſe
b. 18. 3. DE touches the circle ABC, and DCA cuts it,
c. 36. 3. the rectangle AD, DC ſhall be equal [c] to the ſquare of DE. but the rectangle AD, DC is, by Hypotheſis, equal to the ſquare of DB; therefor the ſquare of DE is equal to the ſquare of DB, and the ſtraight line DE equal to the ſtraight line DB. and FE is equal to FB, wherefor DE, EF are equal to DB, BF; and the baſe FD is common to the two triangles DEF, DBF; there-



d. 8. 1. for the angle DEF is equal [d] to the angle DBF, but DEF is a right angle, therefor alſo DBF is a right angle. and FB, if produced, is a diameter, and the ſtraight line which is drawn at right angles to a dia-
e. 16. 3. meter, from the extremity of it, touches [e] the circle. therefor DB touches the circle ABC. wherefor if from a point &c. Q. E. D.

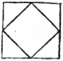
THE ELEMENTS OF EUCLID.

BOOK IV.

DEFINITIONS.

I.

A Rectilineal figure is ſaid to be inſcribed in another rectilineal figure, when all the angles of the inſcribed figure are upon the ſides of the figure in which it is inſcribed, each upon each.

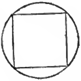


II.

In like manner a figure is ſaid to be deſcribed about another figure, when all the ſides of the circumſcribed figure paſs thro' the angular points of the figure about which it is deſcribed, each thro' each.

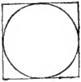
III.

A rectilineal figure is ſaid to be inſcribed in a circle, when all the angles of the inſcribed figure are upon the circumference of the circle.



IV.

A rectilineal figure is ſaid to be deſcribed about a circle, when each ſide of the circumſcribed figure touches the circumference of the circle.

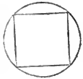


V.

In like manner a circle is ſaid to be inſcribed in a rectilineal figure, when the circumference of the circle touches each ſide of the figure.

VI.

A circle is ſaid to be deſcribed about a rectilineal figure, when the circumference of the circle paſſes thro' all the angular points of the figure about which it is deſcribed.



VII.

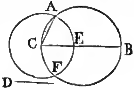
A ſtraight line is ſaid to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

TO place a ſtraight line in a given circle, equal to a given ſtraight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given ſtraight line, not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a ſtraight line BC is placed equal to D. but if it is not, BC is greater than D; make a. 3. 1. CE equal [a] to D, and from the centre C, at the diſtance CE deſcribe the circle AEF, and join CA. therefor becauſe C is the centre of the circle AEF, CA ſhall be equal to CE; but D is equal to CE, therefor D is equal to CA. wherefor in the circle ABC a ſtraight line is placed equal to the given ſtraight line D, which is not greater than the diameter of the circle. Which was to be done.



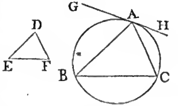
PROP. II. PROB.

IN a given circle to inſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inſcribe in the circle ABC a triangle equiangular to the triangle DEF.

Draw [a] the ſtraight line GAH touching the circle in the point A, a. 11. 1.
and at the point A, in the ſtraight line AH, make [b] the angle HAC equal to the angle DEF; and at the point A, in the ſtraight line AG, make the angle GAB equal to the angle DFE, and join BC. therefor becauſe HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC ſhall be equal [c] to the angle ABC in the alternate ſegment of the circle. but HAC is equal to the angle DEF, therefor alſo the angle ABC is equal to DEF. by the ſame reaſon the angle ACB is equal to the angle DFE; therefor the remaining angle BAC is equal [d] to the remaining angle EDF. wherefor the triangle ABC is equiangu-

b. 23. 1.
c. 32. 3.
d. 32. 1.



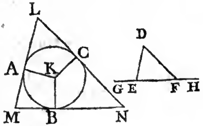
lar to the triangle DEF, and it is inſcribed in the circle ABC. Which was to be done.

PROP. III. PROB.

ABOUT a given circle to deſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to deſcribe a triangle about the circle ABC equiangular to the triangle DEF.

Produce EF both ways to the points G, H, and find the centre K of the circle ABC, and from it draw any ſtraight line KB; at the point
a. 23. 1. K in the ſtraight line KB, make [a] the angle BKA equal to the angle DEG, and the angle BKC equal to the angle DFH; and thro' the points A, B, C draw the ſtraight lines LAM, MBN, NCL touch-
b. 11. 1. ing [b] the circle ABC. therefor becauſe LM, MN, NL touch the circle ABC in the points A, B, C to which from the centre are drawn KA, KB, KC, the angles at the points A, B, C ſhall be right
c. 18. 3. [c] angles. and becauſe the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles; and that two of them KAM, KBM are right angles, the other two AKB, AMB ſhall be equal to two right angles. but the angles DEG,
d. 13. 1. DEF are likewiſe equal [d] to two right angles; therefor the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefor the remaining angle AMB is equal to the remaining



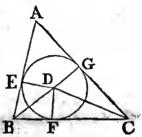
remaining angle DEF. in like manner the angle LNM ſhall be demonſtrated to be equal to DFE; and therefor the remaining angle
MLN ſhall be equal [e] to the remaining angle EDF. wherefor the e. 32. 1.
triangle LMN is equiangular to the triangle DEF. and it is deſcribed about the circle ABC. Which was to be done.

PROP. IV. PROB.

TO inſcribe a circle in a given triangle.

Let the given triangle be ABC; it is required to inſcribe a circle in ABC.

Biſſect [a] the angles ABC, BCA by the ſtraight lines BD, CD meet- a. 9. 1.
ing one another in the point D, from which draw [b] DE, DF, DG b. 12. 1.
perpendicular to AB, BC, CA. and becauſe the angle EBD is equal to the angle FBD, for the angle ABC is biſſected by BD, and that the right angle BED is equal to the right angle BFD, the two triangles EBD, FBD have two angles of the one equal to two angles of the other, and the ſide BD, which is oppoſite to one of the equal angles
in each, is common to both; therefor their other ſides ſhall be equal[c], c. 26. 1.
and DE ſhall be equal to DF. by the ſame reaſon, DG ſhall be equal to DF; therefor the three ſtraight lines DE, DF, DG are equal to one another, and the circle deſcribed from the centre D, at the diſtance of any of them, ſhall paſs thro' the extremities of the other two, and ſhall touch the ſtraight lines AB, BC, CA, becauſe the angles at the points E, F, G are right angles, and the ſtraight line which is drawn from
the extremity of a diameter at right angles to it, touches [d] the circle. d. 16. 3.
therefor



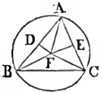
therefor the ſtraight lines AB, BC, CA do each of them touch the circle, and the circle EFG ſhall be inſcribed in the triangle ABC. Which was to be done.

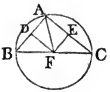
PROP. V. PROB.

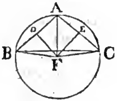
TO deſcribe a circle about a given triangle.

Let the given triangle be ABC; it is required to deſcribe a circle about ABC.

a. 10. 1. Biſſect [a] AB, AC in the points D, E, and from theſe points draw
b. 11. 1. DF, EF at right angles [b] to AB, AC; DF, EF produced ſhall meet one another. for if they do not meet they ſhall be parallel, wherefor AB, AC which are at right angles to them ſhall be parallel; which is abſurd. let them meet in F, and join FA; alſo, if the point F be not in BC, join BF, CF. then becauſe AD is equal to DB, and DF common,
c. 4. 1. and at right angles to AB, the baſe AF ſhall be equal [c] to the baſe FB. in like manner CF ſhall be equal to FA; and therefor BF is equal to FC; and FA, FB, FC ſhall be equal to one another. wherefor the circle deſcribed from the centre F, at the diſtance of one of them, ſhall paſs thro' the extremities of the other two; and ſhall be deſcribed about the triangle ABC. Which was to be done.







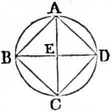
Cor. And it is manifeſt that when the centre of the circle falls within the triangle, each of its angles is leſs than a right angle, each of them being in a ſegment greater than a ſemicircle. but when the centre is in one of the ſides of the triangle, the angle oppoſite to this ſide, being in a ſemicircle, is a right angle. and if the centre falls without the triangle, the angle oppoſite to that ſide without which it is, being in a ſegment leſs than a ſemicircle, is greater than a right angle. Wherefor, if the given triangle be acute angled, the centre of the circle ſhall fall within it; if it be a right angled triangle, the centre ſhall be in the ſide oppoſite to the right angle; and if it be an obtuſe angled triangle, the centre ſhall fall without the triangle, without the ſide oppoſite to the obtuſe angle.

PROP. VI. PROB.

TO inſcribe a ſquare in a given circle.

Let ABCD be the given circle; it is required to inſcribe a ſquare into ABCD.

Draw the diameters AC, BD at right angles to one another; and join AB, BC, CD, DA. becauſe BE is equal to ED, for E is the centre, and that EA is common, and at right angles to BD; the baſe BA ſhall be equal [a] to the baſe AD. and by the ſame reaſon, BC, CD are each of them equal to BA or AD; therefor the quadrilateral figure ABCD is equilateral. It is alſo rectangular; for becauſe the ſtraight line BD is the diameter of the circle ABCD, BAD is a ſemicircle; wherefor the angle BAD is a right [b] angle. by the ſame reaſon each of the angles ABC, BCD, CDA is a right angle. therefor the quadrilateral figure ABCD is rectangular. and it has been ſhewn



a. 4. 1.

b. 31. 3.

Q to

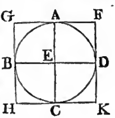
to be equilateral, therefor it is a ſquare; and it is inſcribed in the circle ABCD. Which was to be done.

PROP. VII. PROB.

TO deſcribe a ſquare about a given circle.

Let ABCD be the given circle; it is required to deſcribe a ſquare about it.

Draw two diameters AC, BD of the circle ABCD, at right angles
a. 11. 1. to one another, and thro' the points A, B, C, D draw [a] FG, GH,
HK, KF touching the circle. and becauſe FG touches the circle
ABCD, and EA is drawn from the centre E to the point of
b. 18. 3. contact A, the angles at A ſhall be right [b] angles. by the ſame reaſon,
the angles at the points B, C, D are right angles.
and becauſe the angle AEB is a right angle, as
c. 28. 1. likewiſe is EBG, GH ſhall be parallel [c] to AC.
by the ſame reaſon, AC is parallel to FK. and
in the like manner GF, HK ſhall each of them
be demonſtrated to be parallel to BED. therefor
the figures GK, GC, AK, FB, BK are
d. 34. 1. parallelograms, and GF is therefor equal [d] to HK, and GH to FK. and becauſe AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK; GH, FK ſhall each of them be equal to GF or HK. therefor the quadrilateral figure FGHK is equilateral. It is alſo rectangular; for becauſe GBEA is a parallelogram, and AEB a right angle, AGB ſhall [d] likewiſe be a right angle. in the ſame manner it ſhall be ſhewn that the angles at H, K, F are right angles. therefor the quadrilateral figure FGHK is rectangular. and it was demonſtrated that it is equilateral; therefor it



is a ſquare; and it is deſcribed about the circle ABCD. Which was to be done.

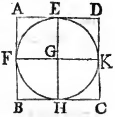
PROP. VIII. PROB.

TO inſcribe a circle in a given ſquare.

Let ABCD be the given ſquare; it is required to inſcribe a circle in ABCD.

Biſſect [a] each of the ſides AB, AD, in the points F, E, and thro' E draw [b] EH parallel to AB or DC, and thro' F draw FK parallel to AD or BC. therefor each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their oppoſite ſides are equal[c]. and becauſe AD is equal to AB, and that AE is the half of AD, and AF the half of AB; AE ſhall be equal to AF. wherefor the ſides oppoſite to theſe are equal, viz. FG to GE. in the ſame manner it ſhall be demonſtrated that GH, GK are each of them equal to FG or GE. therefor the four ſtraight lines GE, GF, GH, GK are equal to one another; and the circle deſcribed from the centre G, at the diſtance of one of them ſhall paſs thro' the extremities of the other three, and ſhall touch the ſtraight lines AB, BC, CD, DA; becauſe the angles at the points E, F, H, K are right [d] angles, and that the ſtraight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle[e]. therefor each of the ſtraight lines AB, BC, CD, DA touches the circle, which therefor is inſcribed in the ſquare ABCD. Which was to be done.

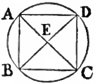
a. 10. 1.
b. 31. 1.
c. 34. 1.
d. 29. 1.
e. 16. 3.



PROP. IX. PROB.

TO deſcribe a circle about a given ſquare.

Let ABCD be the given ſquare; it is required to deſcribe a circle about it.

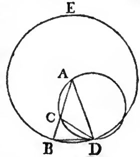


Join AC, BD cutting one another in E. and becauſe DA is equal
to AB, and AC common to the triangles DAC, BAC, the two ſides
DA, AC are equal to the two BA, AC; and the
baſe DC is equal to the baſe BC; wherefor the
a. 8. 1. angle DAC is equal [a] to the angle BAC, and the
angle DAB is biſſected by the ſtraight line AC. in
the ſame manner it ſhall be demonſtrated that each
of the angles ABC, BCD, CDA is biſſected by the
ſtraight lines AC, BD. therefor becauſe the angle DAB is equal to
the angle ABC, and that the angle EAB is the half of DAB, and
EBA the half of ABC, the angle EAB ſhall be equal to the angle
b. 6. 1. EBA; wherefor the ſide EA ſhall be equal [b] to the ſide EB. in
the ſame manner it ſhall be demonſtrated that the ſtraight lines EC,
ED are each of them equal to EA or EB. therefor the four ſtraight
lines EA, EB, EC, ED are equal to one another; and the circle deſcribed
from the centre E, at the diſtance of one of them, ſhall paſs
thro' the extremities of the other three, and ſhall be deſcribed about
the ſquare ABCD. Which was to be done.

PROP. X. PROB.

TO deſcribe an Iſoſceles triangle, each of whoſe angles at the baſe ſhall be double of the third angle.

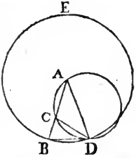
Take any ſtraight line AB, and divide [a] it in the point C, ſo that
the rectangle AB, BC be equal to the ſquare of CA; and from the a. 11. 2.
centre A, at the diſtance AB deſcribe the circle BDE, in which place
[b] the ſtraight line BD equal to AC, which is not greater than the diameter b. 1. 4.
of the circle BDE; join DA, DC, and about the triangle ADC
deſcribe [c] the circle ACD. the triangle ABD ſhall be ſuch as is required, c. 5. 4.
that is, each of the angles ABD, ADB ſhall be double of the
angle BAD.



Becauſe the rectangle AB, BC is equal to the ſquare of AC, and
that AC is equal to BD, the rectangle AB, BC ſhall be equal to the
ſquare of BD. and becauſe from the point
B without the circle ACD two ſtraight lines
BCA, BD are drawn to the circumference,
one of which cuts, and the other meets the
circle, and that the rectangle AB, BC contained
by the whole of the cutting line, and
the part of it without the circle, is equal to
the ſquare of BD which meets it; the ſtraight
line BD ſhall touch [d] the circle ACD. and d. 37. 3.
becauſe BD touches the circle, and DC is
drawn from the point of contact D, the angle BDC ſhall be equal [e] to e. 32. 3.
the angle DAC in the alternate ſegment of the circle; to each of
theſe add the angle CDA, and the whole angle BDA ſhall be equal
to the two angles CDA, DAC. but the exterior angle BCD is equal [f] f. 32. 1.
to the angles CDA, DAC; therefor alſo BDA is equal to BCD. but
BDA is equal [g] to the angle CBD, becauſe the ſide AD is equal g. 5. 1.
to the ſide AB; therefor CBD, i. e. DBA is equal to BCD; and the
three angles BDA, DBA, BCD ſhall be equal to one another. and
becauſe the angle DBC is equal to the angle BCD, the ſide BD ſhall
be

be equal [h] to the ſide DC. but BD was made equal to CA, therefor alſo CA is equal to CD, and the angle CDA equal [g] to the angle DAC. therefor the angles CDA, DAC together, are double of the angle DAC. but BCD is equal to the angles CDA, DAC; therefor alſo BCD is double of DAC. and BCD is equal to each of the angles BDA, DBA; each therefor of the angles BDA, DBA is double of the angle DAB. wherefor an Iſoſceles triangle ABD is deſcribed each of whoſe angles at the baſe is double of the third angle. Which was to be done.

h. 6. 1.
g. 5. 1.



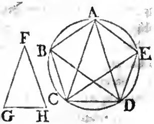
PROP. XI. PROB.

TO inſcribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle, it is required to inſcribe an equilateral and equiangular pentagon in the circle ABCDE.

a. 10. 4. Deſcribe [a] an Iſoſceles triangle FGH having each of the angles at G, H double of the angle at F; and in

b. 2. 4. the circle ABCDE inſcribe [b] the triangle ACD equiangular to the triangle FGH, ſo that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefor each of the angles ACD, CDA



c. 9. 1. ſhall be double of the angle CAD. Biſſect [c] the angles ACD, CDA by

the ſtraight lines CE, DB, and join AB, BC, DE, EA. ABCDE
ſhall be the pentagon required.

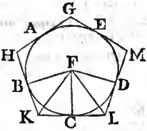
Becauſe each of the angles ACD, CDA is double of CAD, and are biſſected by the ſtraight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another. but equal angles ſtand upon equal [d] circumferences; therefor the five circumferences AB, BC, d. 26. 3.
CD, DE, EA are equal to one another. and equal circumferences are ſubtended by equal [e] ſtraight lines; therefor the five ſtraight lines AB, e. 29. 3.
BC, CD, DE, EA are equal to one another. Wherefor the pentagon ABCDE is equilateral. It is alſo equiangular; for becauſe the circumference AB is equal to the circumference DE, if to each be added BCD, the whole ABCD ſhall be equal to the whole EDCB. and the angle AED ſtands on the circumference ABCD, and the angle BAE on the circumference EDCB; therefor the angle BAE is equal [f] to the angle AED. by the ſame reaſon, each of the angles f. 27. 3.
ABC, BCD, CDE is equal to the angle BAE or AED. therefor the pentagon ABCDE is equiangular; and it has been ſhewn that it is equilateral. Wherefor in the given circle an equilateral and equiangular pentagon has been inſcribed. Which was to be done.

PROP. XII. PROB.

TO deſcribe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle; it is required to deſcribe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon inſcribed in the circle, by the laſt propoſition, be in the points A, B, C, D, E, ſo that the circumferences AB, BC, CD, DE, EA are equal [a]; and thro' the points A, B, a. 11. 4.
C, D,



C, D, E draw GH, HK, KL, LM, MG touching [b] the circle; take b. 11. 1. the centre F, and join FB, FK, FC, FL, FD. and becauſe the ſtraight line KL touches the circle ABCDE in the point C, to which FC is c. 18. 3. drawn from the centre F, FC ſhall be perpendicular [c] to KL; therefor each of the angles at C is a right angle. by the ſame reaſon, the angles at the points B, D are right angles. and becauſe FCK is a right d. 47. 1. angle, the ſquare of FK is equal [d] to the ſquares of FC, CK. by the ſame reaſon the ſquare of FK is equal to the ſquares of FB, BK. therefor the ſquares of FC, CK are equal to the ſquares of FB, BK, of which the ſquare of FC is equal to the ſquare of FB; the remaining ſquare of CK is therefor equal to the remaining ſquare of BK, and the ſtraight line CK equal to BK. and becauſe FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK; and the baſe BK is equal to the baſe e. 8. 1. KC; therefor the angle BFK is equal [e] to the angle KFC, and the angle BKF to FKC. wherefor the angle BFC is double of the angle KFC, and BKC double of FKC. by the ſame reaſon, the angle CFD is double of the angle CFL, and CLD double of CLF. and becauſe the circumference BC is equal to the circumference CD, the angle BFC f. 27. 3. ſhall be equal [f] to the angle CFD. and BFC is double of the angle KFC, and CFD double of CFL; therefor the angle KFC is equal to the angle CFL; and the right angle FCK is equal to the right angle FCL. therefor in the two triangles FKC, FLC, there are two angles of one equal to two angles of the other, each to each, and the ſide FC, which is adjacent to the equal angles in each, is common to both; g. 26. 1. therefor the other ſides ſhall be equal [g] to the other ſides, and the third angle to the third angle. therefor the ſtraight line KC is equal to CL, and the angle FKC to the angle FLC. and becauſe KC is equal to CL, KL ſhall be double of KC. by the ſame reaſon, it ſhall be ſhewn that HK is double of BK. and becauſe BK is equal to KC, as was demonſtrated, and that KL is double of KC, and HK double of BK, HK ſhall be equal to KL. in the ſame manner it ſhall be ſhewn that GH, GM, ML are each of them equal to HK or KL. therefor the pentagon GHKLM is equilateral. It is alſo equiangular; for becauſe the angle FKC is equal to the angle FLC, and that the angle HKL is double of the angle FKC, and KLM double of FLC, as was before demonſtrated, the angle HKL ſhall be equal to KLM. and by the ſame reaſon, each of the angles KHG, HGM, GML is equal to the angle HKL or KLM. therefor the five angles GHK, HKL, KLM, LMG, MGH being equal to one another, the pentagon GHKLM is equiangular. and it is equilateral, as was demonſtrated; and it is deſcribed about the circle ABCDE. Which was to be done.

PROP. XIII. PROB.

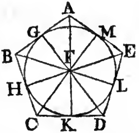
TO inſcribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inſcribe a circle in the pentagon ABCDE.

Biſect [a] the angles BCD, CDE by the ſtraight lines CF, DF, and from the point F in which they meet draw the ſtraight lines FB, FA, FE. therefor becauſe BC is equal to CD, and CF common to the triangles BCF, DCF, the two ſides BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF; therefor the baſe BF is equal [b] to the baſe FD, and the other angles to the other

a. 9. 1.
b. 4. 1.

Book IV. angles, to which the equal ſides are oppoſite; therefor the angle CBF is equal to the angle CDF. and becauſe the angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF, likewiſe CBA ſhall be double of the angle CBF; therefor the angle ABF is equal to the angle CBF; wherefor the angle ABC is biſſected by the ſtraight line BF. in the ſame manner it ſhall be demonſtrated that the angles BAE, AED are biſſected by the ſtraight lines AF, FE. from the point F c. 12. 1. draw [c] FG, FH, FK, FL, FM perpendicular to the ſtraight lines AB, BC, CD, DE, EA. and becauſe the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC; in the triangles FHC, FKC there are two angles of one equal to two angles of the other; and the ſide FC, which is oppoſite to one of the equal angles in each, is common to d. 26. 1. both; therefor the other ſides ſhall be equal[d], each to each; wherefor the perpendicular FH is equal to the perpendicular FK. in the ſame manner it ſhall be demonſtrated that FL, FM, FG are each of them equal to FH or FK; therefor the five ſtraight lines FG, FH, FK, FL, FM are equal to one another. wherefor the circle deſcribed from the centre F, at the diſtance of one of theſe five, ſhall paſs thro' the extremities of the other four, and ſhall touch the ſtraight lines AB, BC, CD, DE, EA, becauſe the angles at the points G, H, K, L, M are right angles; and that a ſtraight line drawn from the extremity of e. 16. 3. the diameter of a circle at right angles to it, touches [e] the circle. therefor each of the ſtraight lines AB, BC, CD, DE, EA touches the circle, and it ſhall be inſcribed in the pentagon ABCDE. Which was to be done.



PROP. XIV.

PROP. XIV. PROB.

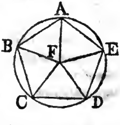
TO deſcribe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to deſcribe a circle about it.

Biſſect [a] the angles BCD, CDE by the ſtraight lines CF, FD, and from the point F in which they meet draw the ſtraight lines FB, FA, FE to the points B, A, E. It ſhall be demonſtrated, in the ſame manner as in the preceeding Propoſition, that the angles CBA, BAE, AED are biſſected by the ſtraight lines FB, FA, FE. and becauſe the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD ſhall be equal to FDC; wherefor the ſide CF ſhall be equal [b] to the ſide FD. in like manner it ſhall be demonſtrated hat FB, FA, FE are each of them equal to FC or FD. therefor the five ſtraight lines FA, FB, FC, FD, FE are equal to one another; and the circle deſcribed from the centre F, at the diſtance of one of them, ſhall paſs thro' the extremities of the other four, and be deſcribed about the equilateral and equiangular pentagon ABCDE. Which was to be done.

a. 9. 1.

b. 6. 1.



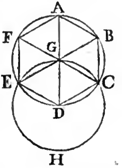
PROP. XV. PROB.

TO inſcribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inſcribe an equilateral and equiangular hexagon in it.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D as a centre, at the diſtance DG deſcribe the circle EGCH, join EG, CG and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA. the hexagon ABCDEF ſhall be equilateral and equiangular.

Becauſe G is the centre of the circle ABCDEF, GE ſhall be equal to GD. and becauſe D is the centre of the circle EGCH, DE ſhall be equal to DG; wherefor GE is equal to ED, and the triangle EGD ſhall be equilateral, and its three angles EGD, GDE, DEG ſhall be equal to one another, becauſe the angles at the
a. 5. 1. baſe of an Iſoſceles triangle are equal[a]. and the
b. 32. 1. three angles of a triangle are equal [b] to two right angles; therefor the angle EGD is the third part of two right angles. in the ſame manner it ſhall be demonſtrated that the angle DGC is alſo the third part of two right angles. and becauſe the ſtraight line GC makes with EB the adjacent angles EGC,
c. 13. 1. CGB equal [c] to two right angles, the remaining angle CGB ſhall be the third part of two right angles; therefor the angles EGD, DGC,
d. 15. 1. CGB are equal to one another. and to theſe are equal [d] the vertical oppoſite angles BGA, AGF, FGE. therefor the ſix angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another. but equal angles
e. 26. 3. ſtand upon equal [e] circumferences; therefor the ſix circumferences AB, BC, CD, DE, EF, FA are equal to one another. and equal circum-
f. 29. 3. ferences are ſubtended by equal [f] ſtraight lines; therefor the ſix ſtraight lines are equal to one another, and the hexagon ABCDEF is equilateral.



equilateral. It is alſo equiangular; for becauſe the circumference AF is equal to ED, to each of theſe add the circumference ABCD, and the whole circumference FABCD ſhall be equal to the whole EDCBA. and the angle FED ſtands upon the circumference FABCD, and the angle AFE upon EDCBA; therefor the angle AFE is equal to FED. in the ſame manner ſhall be demonſtrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED. therefor the hexagon is equiangular. and it is equilateral, as was ſhewn; and it is inſcribed in the given circle ABCDEF. Which was to be done.

Cor. From this it is manifeſt, that the ſide of the hexagon is equal to the ſtraight line from the centre, i. e. to the ſemidiameter of the circle.

And if thro' the points A, B, C, D, E, F there be drawn ſtraight lines touching the circle, an equilateral and equiangular hexagon ſhall be deſcribed about it, which may be demonſtrated from what has been ſaid of the pentagon; and likewiſe a circle may be inſcribed in a given equilateral and equiangular hexagon, and circumſcribed about it, by the ſame method that was uſed for the pentagon.

PROP. XVI. PROB.

To inſcribe an equilateral and equiangular quindecagon in a given circle.

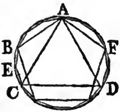
Let ABCD be the given circle; it is required to inſcribe an equilateral and equiangular quindecagon in the circle ABCD.

Let AC be the ſide of an equilateral triangle inſcribed [a] in the circle, and AB the ſide of an equilateral and equiangular pentagon inſcribed [b] in the ſame; therefor of ſuch equal parts as the whole circumference

a. 2. 4.

b. 11. 4.

cumference ABCDF contains fifteen, the circumference ABC, being
the third part of the whole, ſhall contain five; and the circumference
AB, which is the fifth part of the whole, ſhall
contain three; therefor BC their difference ſhall
c. 30. 3. contain two of the ſame parts. biſſect [c] BC in
E, and BE, EC ſhall each of them be the fifteenth
part of the whole circumference ABCD.
therefor if the ſtraight lines BE, EC be drawn,
d. 1. 4. and ſtraight lines equal to them be placed [d]
around in the whole circle, an equilateral and equiangular quindecagon
ſhall be inſcribed in it. Which was to be done.



And in the ſame manner as was done in the pentagon, if thro' the points of diviſion made by inſcribing the quindecagon, ſtraight lines be drawn touching the circle, an equilateral and equiangular quindecagon ſhall be deſcribed about it. and likewiſe, as in the pentagon, a circle may be inſcribed in a given equilateral and equiangular quindecagon, and circumſcribed about it.

THE ELEMENTS OF EUCLID.

BOOK V.

DEFINITIONS.

I.

A Leſſer magnitude is ſaid to be a part of a greater magnitude, when the leſſer meaſures the greater, i. e. 'when the leſſer is 'contained a certain number of times exactly in the greater.'

II.

A greater magnitude is ſaid to be a Multiple of a leſſer, when the greater is meaſured by the leſſer, i. e. 'when the greater contains 'the leſſer a certain number of times exactly.'

III.

" Ratio is a mutual relation of two magnitudes of the ſame kind to " one another, in reſpect of quantity."

IV.

Theſe magnitudes are ſaid to have a ratio to one another, the leſſer of which can be multiplied ſo as to exceed the other.

The

V.

The firſt of four magnitudes is ſaid to have the ſame ratio to the ſecond, which the third has to the fourth, when any equimultiples whatſoever of the firſt and third being taken, and any equimultiples whatſoever of the ſecond and fourth; if the multiple of the firſt be leſs than that of the ſecond, the multiple of the third is alſo leſs than that of the fourth; or, if the multiple of the firſt be equal to that of the ſecond, the multiple of the third is alſo equal to that of the fourth; or, if the multiple of the firſt be greater than that of the ſecond, the multiple of the third is alſo greater than that of the fourth.

VI.

Magnitudes which have the ſame ratio are called Proportionals. N. B. ‘ When four magnitudes are proportionals, it is uſually expreſſed by ‘ ſaying, the firſt is to the ſecond, as the third to the fourth.’

VII.

When of the equimultiples of four magnitudes (taken as in the 5th Definition) the multiple of the firſt is greater than that of the ſecond, but the multiple of the third is not greater than the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third magnitude has to the fourth; and on the contrary, the third is ſaid to have to the fourth a leſſer ratio than the firſt has to the ſecond.

VIII.

“ Analogy, or proportion, is the ſimilitude of ratios.”

IX.

Proportion conſiſts in three terms at leaſt.

X.

When three magnitudes are proportionals, the firſt is ſaid to have to the third the Duplicate ratio of that which it has to the ſecond.

When

XI.

When four magnitudes are continual proportionals, the firſt is ſaid to have to the fourth the Triplicate ratio of that which it has to the ſecond, and ſo forward Quadruplicate &c. increaſing the denomination ſtill by unity, in any number of proportionals.

Definition A, to wit of Compound ratio.

When there are any number of magnitudes of the ſame kind, the firſt is ſaid to have to the laſt of them the ratio compounded of the ratio which the firſt has to the ſecond, and of the ratio which the ſecond has to the third, and of the ratio which the third has to the fourth, and ſo forward unto the laſt magnitude.

For example, if A, B, C, D be four magnitudes of the ſame kind, the firſt A is ſaid to have to the laſt D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is ſaid to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B, the ſame ratio which E has to F; and B to C, the ſame ratio that G has to H; and C to D, the ſame that K has to L; then, by this Definition, A is ſaid to have to D the ratio compounded of ratios which are the ſame with the ratios of E to F, G to H, and K to L. and the ſame thing is to be underſtood when it is more briefly expreſſed by ſaying A has to D, the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the ſame things being ſuppoſed, if M has to N the ſame ratio which A has to D, then, for ſhortneſs ſake, M is ſaid to have to N, the ratio compounded of the ratios of E to F, G to H, and K to L.

XII.

In proportionals, the antecedent terms are called Homologous to one another, as alſo the conſequents to one another.

'Geometers make uſe of the following technical words to ſignify certain ways of changing either the order or magnitude of proportionals, ſo as to continue ſtill to be proportionals.'

XIII.

Permutando, or Alternando, by Permutation, or alternately; this word is uſed when there are four proportionals, and it is inferred, that the firſt has the ſame ratio to the third, which the ſecond has to the fourth; or that the firſt is unto the third, as the ſecond to the fourth. as is ſhewn in the 16th Prop. of this 5th Book.

XIV.

Invertendo, by Inverſion; when there are four proportionals, and it is inferred, that the ſecond is unto the firſt, as the fourth to the third. Prop. B. Book 5th.

XV.

Componendo, by Compoſition; when there are four proportionals, and it is inferred, that the firſt together with the ſecond, is to the ſecond, as the third together with the fourth, is to the fourth. 18th Prop. Book 5th.

XVI.

Dividendo, by Diviſion; when there are four proportionals, and it is inferred, that the Exceſs of the firſt above the ſecond, is to the ſecond, as the Exceſs of the third above the fourth, is to the fourth. 17th Prop. Book 5th.

XVII.

Convertendo, by Converſion; when there are four proportionals, and it is inferred, that the firſt is to its Exceſs above the ſecond, as the third to its Exceſs above the fourth. Prop. E. Book 5th.

XVIII.

Ex aequali (ſc. diſtantia), from equality of diſtance; when there is any number of magnitudes more than two, and as many other ſuch, that they are proportionals when taken two and two of each rank, and it is inferred, that the firſt is to the laſt of the firſt rank of magnitudes, as the firſt is to the laſt of the other. 'of this there are 'the two following kinds, which ariſe from the different order in 'which the magnitudes are taken two and two.'

XIX.

Ex aequali, from equality; this term is uſed ſimply by itſelf, when the firſt magnitude is to the ſecond of the firſt rank, as the firſt to the ſecond of the other rank; and as the ſecond is to the third of the firſt rank, ſo is the ſecond to the third of the other; and ſo forward in order, and the inference is as mentioned in the preceeding Definition; whence this is called Ordinate Proportion. It is demonſtrated in 22d Prop. Book 5th.

XX.

Ex aequali, in proportione perturbata, ſeu inordinata, from equality, in perturbate or diſorderly proportion*; this term is uſed when the firſt magnitude is to the ſecond of the firſt rank, as the laſt ſave one is to the laſt of the ſecond rank; and as the ſecond is to the third of the firſt rank, ſo is the laſt ſave two to the laſt ſave one of the ſecond rank; and as the third is to the fourth of the firſt rank, ſo is the third from the laſt to the laſt ſave two of the ſecond rank; and ſo forwards in a croſs order. and the inference is as in the 18th Definition. It is demonſtrated in 23d Prop. of Book 5th.

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* 4. Prop. Lib. 2. Archimedis de ſphaera et cylindro.

AXIOMS.

I.

EQUIMULTIPLES of the ſame, or of equal magnitudes, are equal to one another.

II.

Theſe magnitudes of which the ſame, or equal magnitudes, are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the ſame multiple of a leſſer.

IV.

That magnitude whoſe multiple is greater than the ſame multiple of another, is greater than that other magnitude.

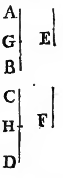
PROP. I. THEOR.

IF any number of magnitudes be equimultiples of as many, each of each; what multiple ſoever any one of them is of its part, the ſame multiple ſhall all the firſt magnitudes be of all the other.

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each; whatſoever multiple AB is of E, the ſame multiple ſhall AB and CD together be of E and F together.

Becauſe AB is the ſame multiple of E that CD is of F, as many magnitudes as are in AB equal to E, ſo many are there in CD equal to F. Divide AB into magnitudes equal to E, viz. AG, GB; and CD into

CH, HD equal each of them to F. the number therefor of the magnitudes CH, HD ſhall be equal to the number of the others AG, GB. and becauſe AG is equal to E, and CH to F; AG and CH together ſhall be equal to [a] E and F together. by the ſame reaſon, becauſe GB is equal to E, and HD to F; GB and HD together ſhall be equal to E and F together. Wherefor as many magnitudes as are in AB equal to E, ſo many are there in AB, CD together equal to E and F together. Therefor whatſoever multiple AB is of E, the ſame multiple is AB and CD together of E and F together.



Therefor if any magnitudes, how many ſoever, be equimultiples of as many, each of each, whatſoever multiple any one of them is of its part, the ſame multiple ſhall all the firſt magnitudes be of all the other. 'for the ſame Demonſtration holds in any 'number of magnitudes, which was here applied to two.' Q. E. D.

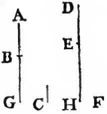
[a] a. Ax. 2. 1.

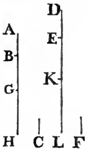
PROP. II. THEOR.

IF the firſt magnitude be the ſame multiple of the ſecond that the third is of the fourth, and the fifth the ſame multiple of the ſecond that the ſixth is of the fourth; then ſhall the firſt together with the fifth be the ſame multiple of the ſecond, that the third together with the ſixth is of the fourth.

Let AB the firſt be the ſame multiple of C the ſecond, that DE the third is of F the fourth; and BG the fifth the ſame multiple of C the ſecond, that EH the ſixth is of F the fourth. Then ſhall AG the firſt together with the fifth be the ſame multiple of C the ſecond, that DH the third together with the ſixth is of F the fourth.

Becauſe AB is the ſame multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F. in like manner, as many as there are in BG equal to C, ſo many are there in EH equal to F. as many then as are in the whole AG equal to C, ſo many are there in the whole DH equal to F. therefor AG is the ſame multiple of C, that DH is of F; that is, AG the firſt and fifth together, is the ſame multiple of the ſecond C, that DH the third and ſixth together is of the fourth F. If therefor the firſt be the ſame multiple &c. Q. E. D.





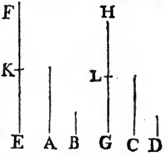
Cor. 'From this it is plain, that if any number of 'magnitudes AB, BG, GH be multiples of another C; 'and as many DE, EK, KL be the ſame multiples 'of F, each of each; the whole of the firſt, viz. AH 'ſhall be the ſame multiple of C, that the whole of 'the laſt, viz. DL is of F.'

PROP. III. THEOR.

IF the firſt be the ſame multiple of the ſecond, which the third is of the fourth; and if of the firſt and third there be taken equimultiples, theſe ſhall be equimultiples the one of the ſecond, and the other of the fourth.

Let A the firſt be the ſame multiple of B the ſecond, that C the third is of D the fourth; and of A, C let the equimultiples EF, GH be taken. EF ſhall be the ſame multiple of B, that GH is of D.

Becauſe EF is the ſame multiple of A, that GH is of C, there ſhall be as many magnitudes in EF equal to A, as are in GH equal to C. let EF be divided into the magnitudes EK, KF, each equal to B, and GH into GL, LH, each equal to C. the number therefor of the magnitudes EK, KF, ſhall be equal to the number of the others GL, LH. and becauſe A is the ſame multiple of B, that C is of D, and that EK is equal to A, and GL to C; EK ſhall be the ſame multiple of B, that GL is of D. by the ſame reaſon KF ſhall be the ſame multiple of B, that LH is of D; and ſo, if there be more parts in EF, GH equal to A, C. becauſe therefor the firſt EK is the ſame multiple of the ſecond B, which the third GL is of the fourth D, and that the fifth KF is the ſame multiple of the ſecond B, which the ſixth LH is of the fourth D; EF the firſt together with the fifth ſhall be the ſame multiple [a] of the ſecond B, which GH the third together with the ſixth is a. 2. 5. of the fourth D. If therefor the firſt &c. Q. E. D.

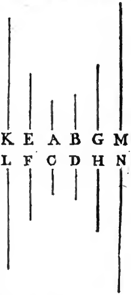


PROP. IV.

PROP. IV. THEOR.

IF the firſt of four magnitudes has to the ſecond the ſame ratio which the third has to the fourth; then ſhall any equimultiples whatever of the firſt and third have the ſame ratio to any equimultiples of the ſecond and fourth, viz. 'the equimultiple of the firſt to that of the ſecond ſhall 'have the ſame ratio, which the equimultiple of the third 'has to that of the fourth.'

Let A the firſt have to B the ſecond, the ſame ratio which the third C has to the fourth D; and of A and C let there be taken any equimultiples whatever E, F; and of B and D any equimultiples whatever G, H. E ſhall have the ſame ratio to G, which F has to H.



Take of E and F any equimultiples whatever K, L, and of G, H, any equimultiples whatever M, N. then becauſe E is the ſame multiple of A, that F is of C; and of E and F have been taken equimultiples K, L; K ſhall a. 3. 5. be the ſame multiple of A, that L is of C[a]. by the ſame reaſon M is the ſame multiple of B, that N is of D. and becauſe as A is to b. Hypoth. B, ſo is C to D[b], and of A and C have been taken certain equimultiples K, L; and of B and D have been taken certain equimultiples M, N; if K be greater than M, L ſhall be greater c. 5. Def. 5. than N; and if equal, equal; if leſs, leſs[c]. And K, L are any equimultiples

tiples whatever of E, F; and M, N any whatever of G, H. as therefor E is to G, ſo is [c] F to H. Therefor if the firſt &c. Q. E. D.

[c] 5. Def. 5.

Cor. Likewiſe if the firſt has the ſame ratio to the ſecond, which the third has to the fourth, then alſo ſhall any equimultiples whatever of the firſt and third have the ſame ratio to the ſecond and fourth. and in like manner the firſt and the third ſhall have the ſame ratio to any equimultiples whatever of the ſecond and fourth.

Let A the firſt have to B the ſecond, the ſame ratio which the third C has to the fourth D, and of A and C let E and F be any equimultiples whatever; E ſhall be to B, as F to D.

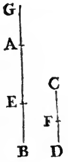
Take of E, F any equimultiples whatever K, L and of B, D any equimultiples whatever G, H; and it ſhall be demonſtrated, as before, that K is the ſame multiple of A, that L is of C. and becauſe A is to B, as C is to D, and of A and C certain equimultiples have been taken, viz. K and L; and of B and D certain equimultiples G, H; if K be greater than G, L ſhall be greater than H; and if equal, equal; if leſs, leſs[c]. and K, L are any equimultiples of E, F, and G, H any whatever of B, D; as therefor E is to B, ſo is F to D. and in the ſame way the other caſe is demonſtrated.

PROP. V. THEOR.

If one magnitude be the ſame multiple of another, which a magnitude taken from the firſt is of a magnitude taken from the other; the remainder ſhall be the ſame multiple of the remainder, that the whole is of the whole.

Let the magnitude AB be the ſame multiple of CD, that AE taken from the firſt, is of CF taken from the other; the remainder EB ſhall be the ſame multiple of the remainder FD, that the whole AB is of the whole CD.

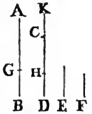
Take AG the ſame multiple of FD, that AE is of CF. therefor a. 1. 5. ſhall [a] AE be the ſame multiple of CF, that EG is of CD. but AE, by the hypotheſis, is the ſame multiple of CF, that AB is of CD. therefor EG is the ſame multiple of CD which AB is of CD; wherefor EG is equal b. 1. Ax. 5. to AB[b]. take from them the common magnitude AE, the remainder AG ſhall be equal to the remainder EB. Wherefor becauſe AE is the ſame multiple of CF, that AG is of FD, and that AG is equal to EB; AE ſhall be the ſame multiple of CF, that EB is of FD. but AE is the ſame multiple of CF, that AB is of CD; therefor EB is the ſame multiple of FD, that AB is of CD. Therefor if one magnitude &c. Q. E. D.



PROP. VI. THEOR.

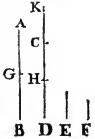
IF two magnitudes be equimultiples of two others, and if equimultiples of theſe others be taken from the firſt two, the remainders ſhall either be equal to theſe others, or be equimultiples of them.

Let the two magnitudes AB, CD be equimultiples of the two E, F, and AG, CH taken from the firſt two be equimultiples of the ſame E, F; the remainders GB, HD ſhall either be equal to E, F, or equimultiples of them.



Firſt, Let GB be equal to E; HD ſhall be equal to F. make CK equal to F; and becauſe AG is the ſame multiple of E, that CH is of F, and that GB is equal to E, and CK to F; AB ſhall be the ſame multiple of E, that KH is of F. But AB, by the hypotheſis, is the ſame multiple of E that CD is of F; therefor KH is the ſame multiple of F, that CD is of F; wherefor KH is equal to CD[a]. take away the common magnitude CH, then the remainder KC is equal to the remainder HD. but KC is equal to F, HD therefor is equal to F. a. 1. Ax. 5.

But let GB be a multiple of E, then ſhall HD be the ſame multiple of F. Make CK the ſame multiple of F, that GB is of E. and becauſe AG is the ſame multiple of E, that CH is of F, and GB the ſame multiple of E, that CK is of F, AB ſhall be the ſame multiple of E, that KH is of F[b]. but AB is the ſame multiple of E, that CD is of F; therefor KH is the ſame multiple of F, that CD is of it; wherefor KH is equal to CD[a]. take away CH from both, the remainder KC ſhall be equal to the remainder HD. and becauſe GB is the ſame multiple of E, that KC is of F, and that KC is equal to HD; therefor HD is the ſame multiple of F, that GB is of E. If therefor two magnitudes &c. Q. E. D. b. 2. 5.



PROP. A. THEOR.

IF the firſt of four magnitudes has to the ſecond, the ſame ratio which the third has to the fourth; if the firſt be greater than the ſecond, the third ſhall be greater than the fourth; and if equal, equal; if leſs, leſs.

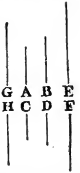
Take any equimultiples of each of them, as the doubles of each. then by Def. 5th of this Book, if the double of the firſt be greater than the double of the ſecond, the double of the third ſhall be greater than the double of the fourth. but if the firſt be greater than the ſecond, the double of the firſt ſhall be greater than the double of the ſecond. wherefor alſo the double of the third ſhall be greater than the double of the fourth; therefor the third ſhall be greater than the fourth. in like manner, if the firſt be equal to the ſecond, or leſs than it, the third ſhall be proved to be equal to the fourth, or leſs than it. therefor if the firſt &c. Q. E. D.

PROP. B. THEOR.

IF four magnitudes are proportionals, they ſhall likewiſe be proportionals when taken inverſely.

If the magnitude A be unto B, as C is to D, then alſo inverſely B ſhall be to A, as D to C.

Take of B and D any equimultiples whatever E and F; and of A and C any equimultiples whatever G and H. Firſt, Let E be greater than G, then ſhall G be leſs than E; and becauſe A is unto B, as C is to D, and of A and C the firſt and third, G and H are equimultiples; and of B and D the ſecond and fourth, E and F are equimultiples; and that G is leſs than E, therefor [a] H is leſs than F; that is, F is greater than H. if therefore E be greater than G, F ſhall be greater than H. in like manner, if E be equal to G, F ſhall be ſhewn to be equal to H; and if leſs, leſs. and E, F are any equimultiples whatever of B and D, and G, H any whatever of A and C. Therefor as B is to A, ſo is D to C. If then four magnitudes &c. Q. E. D.



[a] 5. Def. 5.

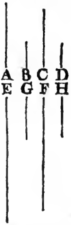
PROP. C.

PROP. C. THEOR.

IF the firſt be the ſame multiple of the ſecond, or the ſame part of it, that the third is of the fourth; the firſt ſhall be to the ſecond, as the third is to the fourth.

Let the firſt A be the ſame multiple of B the ſecond, that C the third is of the fourth D. A ſhall be to B, as C is to D.

Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H. then becauſe A is the ſame multiple of B that C is of D; and that E is the ſame multiple of A, that F is of C; E ſhall be the ſame multiple of B, that F is of D[a]; therefor E and F are the ſame multiples of B and D. but G and H are equimultiples of B and D; therefor if E be a greater multiple of B, than G is of it, F ſhall be a greater multiple of D, than H is of D; that is, if E be greater than G, F ſhall be greater than H. in like manner, if E be equal to G, or leſs than it, F ſhall be equal to H, or leſs than it. But E, F are equimultiples, any whatever, of A, C, and G, H any equimultiples whatever of B, D. Therefor A is to B, as C is to D[b].



a. 3. 5.

b. 5. Def. 5.

But let the firſt A be the ſame part of the ſecond B, that the third C is of the fourth D. A ſhall be to B, as C is to D. for B ſhall be the ſame multiple of A, that D is of C; wherefor by the preceeding caſe B is to A, as D is to C; and inverſely [c] A ſhall be to B, as C is to D. Therefor if the firſt be the ſame multiple &c. Q. E. D.

c. B. 5.

PROP. D.

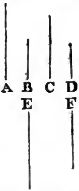
PROP. D. THEOR.

IF the firſt be to the ſecond as the third to the fourth, and if the firſt be a multiple, or part of the ſecond; the third ſhall be the ſame multiple, or the ſame part of the fourth.

Let A be to B, as C is to D; and firſt let A be a multiple of B; C ſhall be the ſame multiple of D.

Take E equal to A, and what multiple A or E is of B, make F the ſame multiple of D. then becauſe A is to B, as C is to D; and of B the ſecond and D the fourth equimultiples have been taken E and F; A ſhall be to E, as C to F[a]. but A is equal to E, therefor C is equal to F[b]. and F is the ſame multiple of D, that A is of B. Wherefor C is the ſame multiple of D, that A is of B.

a. Cor. 4. 5.
b. A. 5.



Next, Let the firſt A be a part of the ſecond B, C the third ſhall be the ſame part of the fourth D. Becauſe A is to B, as C is to D; then, inverſely B ſhall[c] be to A, as D to C. but A is a part of B, therefor B is a multiple of A, and, by the preceding caſe, D ſhall be the ſame multiple of C; that is, C ſhall be the ſame part of D, that A is of B. Therefor if the firſt &c. Q. E. D.

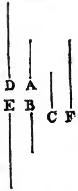
See the Figure at the foot of the preceding Page.
c. B. 5.

PROP. VII. THEOR.

EQUAL magnitudes have the ſame ratio to the ſame magnitude; and the ſame unto equal magnitudes has the ſame ratio.

Let A and B be equal magnitudes, and C any other. A and B ſhall each of them have the ſame ratio to C. and C ſhall have the ſame ratio to each of A and B.

Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F. then becauſe D is the ſame multiple of A, that E is of B, and that A is equal to B, D ſhall be [a] equal to E. therefor if D be greater than F, E ſhall be greater than F; and if equal, equal; if leſs, leſs. and D, E are any equimultiples of A, B, and F is any multiple of C. Therefor [b] as A is to C, ſo is B to C.



a. 1. Ax. 5.

b. 5. Def. 5.

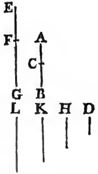
Likewiſe C ſhall have the ſame ratio to A that it has to B. for, having made the ſame conſtruction, in the like manner D ſhall be ſhewn equal to E. therefor if F be greater than D, it ſhall likewiſe be greater than E; and if equal, equal; if leſs, leſs. and F is any multiple whatever of C, and D, E, are any equimultiples whatever of A, B. Therefor C is to A, as C is to B[b]. Therefor equal magnitudes &c. Q. E. D.

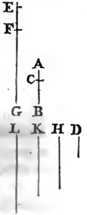
PROP. VIII. THEOR.

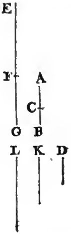
THE greater of two unequal magnitudes has a greater ratio to any magnitude than the leſs has to the ſame. and the ſame magnitude has to the leſſer of two magnitudes a greater ratio than it has to the greater.

Let AB, BC be two unequal magnitudes of which AB is the greater, and let D be any other magnitude. AB ſhall have a greater ratio to D than BC to D. and D ſhall have a greater ratio to BC than unto AB.

If the magnitude which is not the greater of the two AC, CB, be not leſſer than D, take EF, FG the doubles of AC, CB, as in Fig. 1. but if that which is not the greater of the two AC, CB be leſſer than D (as in Fig. 2. and 3.) this magnitude can be multiplied ſo as to become greater than D, whether it be AC or CB. Let it be multiplied until it become greater than D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the ſame multiple of CB. therefore EF and FG are each of them greater than D. and in every one of the caſes take H the double of D, K its triple, and ſo forwards, till the multiple of D is that which firſt becomes greater than FG. let L be that multiple of D which is firſt greater than FG, and K the multiple of D which is next leſſer than L.







Then becauſe L is the multiple of D which is the firſt that becomes greater than FG, the next preceding multiple K ſhall not be greater than FG; that is, FG ſhall not be leſſer than K. and ſince EF is the ſame multiple of AC, that FG is of CB, FG ſhall be the ſame multiple of CB,
a. 1. 5. that EG is of AB[a]; wherefor EG and FG are equimultiples of AB and CB. and it was ſhewn that FG was not leſſer than K, and, by the conſtruction, EF is greater than D;

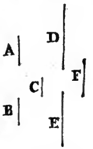
D; therefor the whole EG is greater than K and D together. but K together with D is equal to L; therefor EG is greater than L; but FG is not greater than L; and EG, FG are equimultiples of AB, BC, and L is a multiple of D; therefor [b] AB has to D a greater ratio than BC has to D. b. 7. Def. 5.

Alſo D ſhall have to BC a greater ratio than it has to AB. for, having made the ſame conſtruction, in like manner it ſhall be ſhewn that L is greater than FG, but that it is not greater than EG. and L is a multiple of D; and FG, EG are equimultiples of CB, AB. Therefor D has to CB a greater ratio [b] than it has to AB. Wherefor of two unequal magnitudes &c. Q. E. D.

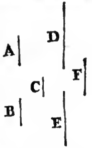
PROP. IX. THEOR.

THESE magnitudes which have the ſame ratio to the ſame magnitude are equal to one another; and theſe to which the ſame magnitude has the ſame ratio are equal to one another.

Let A, B have each of them the ſame ratio to C; A ſhall be equal to B. for if they are not equal, one of them is greater than the other; let A be the greater; then, by what was ſhewn in the preceeding Propoſition, there are ſome equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let theſe multiples be taken, and let D, E, be the equimultiples of A, B, and F the multiple of C ſuch, that D is greater than F, and E is not greater than F. but becauſe A is to C, as B is to C, and



of A, B are taken equimultiples D, E, and of C is taken a multiple F; and that D is greater than F; E ſhall alſo be a. 5. Def. 5. greater than F[a]; but E is not greater than F, which is impoſſible. A therefor and B are not unequal; that is, they are equal.



Next, Let C have the ſame ratio to each of the magnitudes A and B; A ſhall be equal to B. for if they are not, one of them is greater than the other; let A be the greater, therefor, as was ſhewn in Prop. 8th, there is ſome multiple F of C, and ſome equimultiples E and D of B and A ſuch, that F is greater than E, and not greater than D. but becauſe C is to B, as C is to A, and that F the multiple of the firſt is greater than E the multiple of the ſecond; F the multiple of the third ſhall be greater than D the multiple of the fourth[a]. but F is not greater than D, which is impoſſible. Therefor A is equal to B. Wherefor theſe magnitudes &c. Q. E. D.

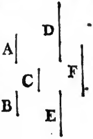
PROP. X. THEOR.

THAT magnitude which has a greater ratio than another has unto the ſame magnitude is the greater of the two. and that magnitude to which the ſame has a greater ratio than it has unto another magnitude is the leſſer of the two.

Let A have to C a greater ratio than B has to C; A ſhall be greater than B. for becauſe A has a greater ratio to C, than B has to a. 7. Def. 5. C, there are [a] ſome equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it. let them be taken, and let D, E be

E be equimultiples of A, B, and F a multiple of C ſuch, that D is greater than F, but E is not greater than F. therefor D is greater than E. and becauſe D and E are equimultiples of A and B, and D is greater than E; A ſhall be [b] greater than B.

Next, Let C have a greater ratio to B than it has to A; B ſhall be leſs than A. for [a] there is ſome multiple F of C, and ſome equimultiples E and D of B and A ſuch, that F is greater than E, but is not greater than D. E therefor is leſs than D; and becauſe E and D are equimultiples of B and A, B ſhall [b] be leſs than A. That magnitude therefor &c. Q. E. D.



b. 4. Ax. 5.

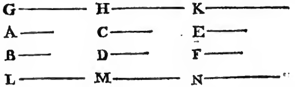
a. 7. Def. 5.

PROP. XI. THEOR.

THESE ratios that are the ſame to the ſame ratio, are the ſame to one another.

Let A be to B, as C is to D; and as C to D, ſo let E be to F; A ſhall be to B, as E to F.

Take of A, C, E any equimultiples whatever G, H, K; and of B, D, F any equimultiples whatever L, M, N. Therefor becauſe as A is to B, ſo is C to D, and of A, C are taken equimultiples G, H; and L, M of B, D; if G be greater than L, H ſhall be greater than M;



a. 5. Def. 5.

See the Figure below.

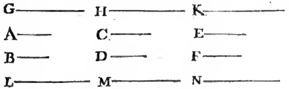
and if equal, equal; and if leſs, leſſer[a]. Again, becauſe C is to D, as E is to F, and H, K are taken equimultiples of C, E; and M, N of D, F; if H be greater than M, K ſhall be greater than N; and if equal, equal; and if leſs, leſſer. but if G be greater than L, it has been ſhewn that H ſhall be greater than M; and if equal, equal; and if leſs, leſſer; therefor if G be greater than L, K ſhall be greater than N; and if equal, equal; and if leſs, leſſer. and G, K, are any equimultiples whatever of A, E; and L, N any whatever of B, F. Therefor as A is to B, ſo is E to F[a]. Wherefor theſe ratios &c. Q. E. D.

PROP. XII. THEOR.

IF any number of magnitudes be proportionals, as one of the antecedents is to its conſequent, ſo ſhall all the antecedents together be to all the conſequents together.

Let any number of magnitudes A, B, C, D, E, F, be proportionals; that is, as A is to B, ſo C to D, and E to F. as A is to B, ſo ſhall A, C, E together be to B, D, F together.

Take of A, C, E any equimultiples whatever G, H, K; and of



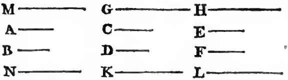
B, D, F any equimultiples whatever L, M, N. becauſe therefor A is to B, as C is to D, and as E to F; and that G, H, K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F; if G be greater than

than L, H ſhall be greater than M, and K greater than N; and if equal, equal; and if leſs, leſſer[a]. Wherefor if G be greater than L, then a. 5. Def. 5.
G, H, K together ſhall be greater than L, M, N together; and if equal, equal; and if leſs, leſſer. and G, and G, H, K together are any equimultiples of A, and A, C, E together, becauſe if there be any number of magnitudes equimultiples of as many, each of each, what multiple one of them is of its part, the ſame multiple is the whole of the whole[b]. by the ſame reaſon L, and L, M, N are any equimul- b. 1. 5.
tiples of B, and B, D, F. as therefor A is to B, ſo are A, C, E together to B, D, F together. Wherefor if any number &c. Q. E. D.

PROP. XIII. THEOR.

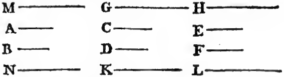
IF the firſt has to the ſecond the ſame ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth has to the ſixth.

Let A the firſt have the ſame ratio to B the ſecond which C the third has to D the fourth, but C the third to D the fourth a greater



ratio than E the fifth to F the ſixth. alſo the firſt A ſhall have to the ſecond B a greater ratio than the fifth E to the ſixth F.

Becauſe C has a greater ratio to D, than E to F, there are ſome equimultiples of C and E, and ſome of D and F ſuch, that the multiple of C is greater than the multiple of D, but the multiple of E is a. 7. Def. 5. not greater than the multiple of F[a]. let theſe be taken, and of C, E let G, H be equimultiples, and K, L equimultiples of D, F ſuch, that G is greater than K, but H not greater than L; and what multiple G is of C take M the ſame multiple of A; and what multiple K is of D, take N the ſame multiple of B. then becauſe A is to B, as C to D, and of A and C, M and G are equimultiples, and of B and D, N and K are equimultiples; if M be greater than N, G ſhall be greater than b. 5. Def. 5. K; and if equal, equal; and if leſs, leſſer[b]; but G is greater than K, therefor M is greater than N. but H is not greater than L; and M, H are equimultiples of A, E; and N, L equimultiples of B, F. Therefor A has a greater ratio to B, than E has to F[a]. Wherefor if the firſt &c. Q. E. D.



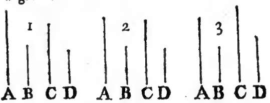
Cor. And if the firſt has a greater ratio to the ſecond, than the third has to the fourth; but the third the ſame ratio to the fourth, which the fifth has to the ſixth; in the like manner it ſhall be demonſtrated that the firſt has a greater ratio to the ſecond than the fifth has to the ſixth.

PROP. XIV. THEOR.

If the firſt has to the ſecond the ſame ratio, which the third has to the fourth; then, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; and if leſs, leſſer.

Let the firſt A have to the ſecond B the ſame ratio, which the third C has to the fourth B; if A be greater than C, B ſhall be greater than D.

Becauſe A is greater than C, and B is any other magnitude, A ſhall have to B a greater ratio than C to B[a]. but as C is to D, ſo is a. 8. 5.



A to B; therefor C ſhall alſo have to D a greater ratio than C has to B[b]. but of two magnitudes, that to which the ſame has the greater b. 13. 5. ratio is the leſſer[c]. wherefor D is leſſer than B; that is, B is greater c. 10. 5. than D.

Secondly, If A be equal to C, B ſhall be equal to D. for becauſe A is to B, as C, that is A, to D; B ſhall be equal to D[d]. d. 9. 5.

Thirdly, If A be leſſer than C, B ſhall be leſſer than D. for C is greater than A, and becauſe C is to D, as A is to B, D ſhall be greater than B by the firſt caſe; wherefor B is leſſer than D. Therefor if the firſt &c. Q. E. D.

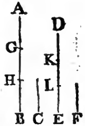
PROP. XV. THEOR.

MAGNITUDES have the ſame ratio to one another which their equimultiples have.

Let AB be the ſame multiple of C that DE is of F. C ſhall be to be F, as AB is to DE.

Becauſe AB is the ſame multiple of C that DE is of F, there ſhall

be as many magnitudes in AB equal to C, as there are in DE equal to F. Let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE. then the number of the firſt AG, GH, HB ſhall be equal to the number of the laſt DK, KL, LE. and becauſe AG, GH, HB are all equal, as alſo DK, KL, LE equal to one another; AG ſhall be to DK, as GH to KL, and as HB to
a. 7. 5. LE[a]. and as one of the antecedents to its conſequent, ſo ſhall all the antecedents together be to all
b. 12. 5. the conſequents together[b]; wherefor as AG is to DK, ſo is AB to DE. but AG is equal to C, and DK to F. therefor as C is to F, ſo ſhall AB be to DE. Therefor any two &c. Q. E. D.

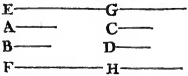


PROP. XVI. THEOR.

IF four magnitudes all of the ſame kind be proportionals, they ſhall alſo be proportionals when taken alternately.

Let the four magnitudes A, B, C, D be proportionals, viz. as A is to B, ſo let C be to D. they ſhall alſo be proportionals when taken alternately; that is, A ſhall be to C, as B to D.

Take of A and B any equimultiples whatever E and F; and of C and D take any equimultiples whatever G and H. and becauſe E is the ſame multiple of A, that F is of B, and that magnitudes have the ſame ratio to one ano-
a. 15. 5. ther which their equimultiples have[a]; A ſhall be to B, as E is to F.



but as A is to B, ſo is C to D. wherefor as C is to D, ſo [b] is E to F. again, becauſe G, H are equimultiples of C, D, as C is to D, ſo is b. 11. 5. G to H[a]; but as C is to D, ſo is E to F. Wherefor as E is to a. 15. 5. F, ſo is G to H[b]. But when four magnitudes are proportionals, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; if leſs, leſſer[c]. Wherefor if E be greater c. 14. 5. than G, likewiſe F ſhall be greater than H; and if equal, equal; if leſs, leſſer. and E, F are any equimultiples whatever of A, B; and G, H any whatever of C, D. Therefor A is to C, as B to D[d]. If then d. 5. Def. 5. four magnitudes &c. Q. E. D.

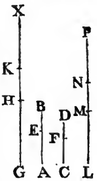
PROP. XVII. THEOR.

IF magnitudes taken jointly be proportionals, they ſhall alſo be proportionals when taken ſeparately, i. e. if two magnitudes together have to one of them, the ſame ratio which two others have to one of them, the remaining one of the firſt two ſhall have to the other, the ſame ratio which the remaining one of the laſt two has to the other of them.

Let AB, BE; CD, DF be the magnitudes taken jointly which are proportionals; that is, as AB to BE, ſo is CD to DF; they ſhall likewiſe be proportionals taken ſeparately, viz. as AE to EB, ſo ſhall CF be to FD.

Take of AE, EB, CF, FD any equimultiples whatever GH, HK, LM, MN; and again, of EB, FD, take any whatever equimultiples KX, NP. and becauſe GH is the ſame multiple of AE that HK is of EB, GH ſhall be the ſame multiple [a] of AE, that GK is of AB. a. 1. 5. but GH is the ſame multiple of AE, that LM is of CF; wherefor

GK is the ſame multiple of AB, that LM is of CF. Again, becauſe LM is the ſame multiple of CF that MN is of FD; LM ſhall be the a. 1. 5. ſame multiple [a] of CF, that LN is of CD. but LM was ſhewn to be the ſame multiple of CF, that GK is of AB; GK therefor is the ſame multiple of AB, that LN is of CD; that is, GK, LN are equimultiples of AB, CD. Next, becauſe HK is the ſame multiple of EB, that MN is of FD; and that KX is alſo the ſame multiple of EB, that NP b. 2. 5. is of FD; HX ſhall be the ſame multiple [b] of EB that MP is of FD. And becauſe AB is to BE, as CD is to DF, and that of AB and CD, GK and LN are equimultiples, and of EB and FD, HX and MP are equimultiples; if GK be greater than HX, then LN c. 5. Def. 5. ſhall be greater than MP; and if equal, equal; and if leſs, leſſer [c]. but if GH be greater than KX, by adding the common part HK to both, GK ſhall be greater than HX; wherefor alſo LN ſhall be greater than MP; and by taking away MN from both, LM ſhall be greater than NP. therefor if GH be greater than KX, LM ſhall be greater than NP. In like manner it ſhall be demonſtrated, that if GH be equal to KX, likewiſe LM ſhall be equal to NP; and if leſs, leſſer. and GH, LM are any equimultiples whatever of AE, CF, and KX, NP are any whatever of EB, FD. Therefor [c] as AE is to EB, ſo is CF to FD. If then magnitudes &c. Q. E. D.



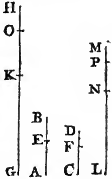
PROP. XVIII.

PROP. XVIII. THEOR.

If magnitudes taken ſeparately be proportionals, they ſhall likewiſe be proportionals when taken jointly, i. e. if the firſt be to the ſecond, as the third to the fourth, likewiſe the firſt and ſecond together ſhall be to the ſecond, as the third and fourth together to the fourth.

Let AE, EB, CF, FD be proportionals; that is, as AE to EB, ſo is CF to FD; they ſhall alſo be proportionals when taken jointly; that is, as AB to BE, ſo ſhall CD be to DF.

Take of AB, BE, CD, DF any equimultiples whatever GH, HK, LM, MN; and again of BE, DF take any whatever equimultiples KO, NP. and becauſe KO, NP are equimultiples of BE, DF; and KH, NM are equimultiples likewiſe of BE, DF, if KO the multiple of BE be greater than KH which is a multiple of the ſame BE, likewiſe NP the multiple of DF ſhall be greater than NM the multiple of the ſame DF; and if KO be equal to KH, NP ſhall be equal to NM; and if leſs, leſſer.

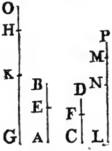


Firſt, Let KO be not greater than KH, therefor NP ſhall not be greater than NM. and becauſe GH, HK are equimultiples of AB, BE, and that AB is greater than BE, GH ſhall be greater * than HK; but KO is not greater than KH, wherefor GH is greater than KO. In like manner it ſhall be ſhewn that LM is greater than NP. Therefor if KO be not greater than KH, then ſhall GH the multiple of AB be ever greater than KO the multiple

* 3. Ax. 5.

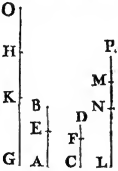
of BE; and likewiſe LM the multiple of CD ſhall be greater than NP the multiple of DF.

Next, Let KO be greater than KH; therefor, as has been ſhewn, NP ſhall be greater than NM. and becauſe the whole GH is the ſame multiple of the whole AB, that HK is of BE, the remainder GK ſhall be the ſame multiple of the remainder AE that GH is of
a. 5. 5. AB[a], which is the ſame that LM is of CD. In like manner, becauſe LM is the ſame multiple of CD, that MN is of DF, the remainder LN ſhall be the ſame multiple of the remainder CF, that the whole LM is of the whole CD[a]. but it was ſhewn that LM is the ſame multiple of CD that GK is of AE; therefor GK is the ſame multiple of AE that LN is of CF; that is, GK, LN are equimultiples of AE, CF. and becauſe KO, NP are equimultiples of BE, DF, if from KO, NP there be taken KH, NM which are likewiſe equimultiples of BE, DF, the remainders HO, MP ſhall either be equal to BE, DF, or equi-
b. 6. 5. multiples of them[b]. Firſt, Let HO, MP be equal to BE, DF; and becauſe AE is to EB, as CF to FD, and that GK, LN are equimul-
c. Cor. 4. 5. tiples of AE, CF; GK ſhall be to EB, as LN to FD[c]. but HO is equal to EB, and MP to FD; wherefor GK is to HO, as LN to MP. If therefor GK be greater than HO, LN ſhall be greater than MP;
d. A. 5. and if equal, equal; and if leſs, leſſer[d].



But let HO, MP be equimultiples of EB, FD; and becauſe AE is to EB, as CF to FD, and that of AE, CF are taken equimultiples GK, LN; and of EB, FD, the equimultiples HO, MP; if GK be greater than HO, LN ſhall be greater than MP; and if equal, equal;
e. 5. Def. 5. and if leſs, leſſer[e]; which was likewiſe ſhewn in the preceeding caſe.

If therefor GH be greater than KO, taking KH from both, GK ſhall be greater than HO; wherefor alſo LN ſhall be greater than MP; and, adding NM to both, LM ſhall be greater than NP. therefor if GH be greater than KO, LM ſhall be greater than NP. In like manner it ſhall be ſhewn that if GH be equal to KO, LM ſhall be equal to NP; and if leſs, leſſer. And in the caſe in which KO is not greater than KH, it has been ſhewn that GH is ever greater than KO, and likewiſe LM than NP. but GH, LM are any equimultiples of AB, CD, and KO, NP are any whatever of BE, DF; therefore [c] as AB is to BE, ſo is CD to DF. If then magnitudes &c. Q. E. D.

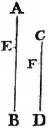


c. 5. Def. 5.

PROP. XIX. THEOR.

IF a whole magnitude be to a whole, as a magnitude taken from the firſt is to a magnitude taken from the other; the remainder ſhall be to the remainder as the whole to the whole.

Let the whole AB be to the whole CD, as AE a magnitude taken from AB to CF a magnitude taken from CD; the remainder EB ſhall be to the remainder FD, as the whole AB, to the whole CD.



Becauſe AB is to CD, as AE to CF; likewiſe alternately[a], BA ſhall be to AE, as DC to CF. and becauſe if magnitudes taken jointly be proportionals, they ſhall alſo be proportionals [b] when taken ſeparately; as therefor

a. 16. 5.

b. 17. 5.

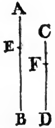
See the Figure below.

for BE is to EA, ſo is DF to FC; and alternately, as BE is to DF, ſo is EA to FC. but as AE to CF, ſo, by the Hypotheſis, is AB to CD; therefor alſo BE the remainder ſhall be to the remainder DF, as the whole AB to the whole CD. Wherefor if the whole &c. Q. E. D.

Cor. If the whole be to the whole, as a magnitude taken from the firſt is to a magnitude taken from the other; likewiſe the remainder ſhall be to the remainder, as the magnitude taken from the firſt to that taken from the other. the Demonſtration is contained in the preceeding.

PROP. E. THEOR.

IF four magnitudes be proportionals, they ſhall alſo be proportionals by converſion, i. e. the firſt ſhall be to its exceſs above the ſecond, as the third to its exceſs above the fourth.



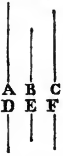
Let AB be to BE, as CD to DF; then BA ſhall be to AE, as DC to CF.

a. 17. 5. Becauſe AB is to BE, as CD to DF, by diviſion[a],
b. B. 5. AE ſhall be to EB, as CF to FD; and by inverſion[b],
BE is to EA, as DF to FC. Wherefor, by Compoſi-
c. 18. 5. tion[c], BA ſhall be to AE, as DC is to CF. If therefor
four &c. Q. E. D.

PROP. XX. THEOR.

IF there be three magnitudes, and other three, which taken two and two have the ſame ratio; if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſſer.

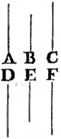
Let A, B, C be three magnitudes, and D, E, F other three, which taken two and two have the ſame ratio, viz. as A is to B, ſo is D to E; and as B to C, ſo is E to F. If A be greater than C, D ſhall be greater than F; and if equal, equal; and if leſs, leſſer.

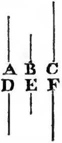


Becauſe A is greater than C, and B is any other magnitude, and that the greater has to the ſame magnitude a greater ratio than the leſſer has to it[a]; a. 8. 5. A ſhall have to B a greater ratio than C has to B. but as D is to E, ſo is A to B; therefor [b] D has to E a greater ratio than C to B. b. 13. 5. and becauſe B is to C, as E to F, by inverſion, C ſhall be to B, as F is to E; and D was ſhewn to have to E a greater ratio than C to B; therefor D has to E a greater ratio than F to E[c]. c. Cor. 13. 5. but the magnitude which has a greater ratio than another to the ſame magnitude, is the greater of the two[d]. d. 10. 5. D is therefor greater than F.

Secondly, Let A be equal to C; D ſhall be equal to F. becauſe A and C are equal to one another, A ſhall be to B, as C is to B[e]. e. 7. 5. but A is to B, as D to E; and C is to B, as F to E; wherefor D is to E, as F to E[f]; f. 11. 5. and therefor D is equal to F[g]. g. 9. 5.

Next, Let A be leſſer than C; D ſhall be leſſer than F. for becauſe C is greater than A, and that, as was ſhewn in the firſt caſe, C is to B, as F to E, and in like manner B is to A, as E to D; F ſhall be greater than D, by the firſt caſe; and therefor D is leſſer than F. Therefor if there be three &c. Q. E. D.



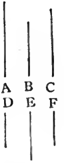


PROP. XXI.

PROP. XXI. THEOR.

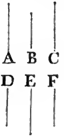
IF there be three magnitudes, and other three, which have the ſame ratio taken two and two, but in a croſs order; if the firſt magnitude be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſſer.

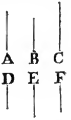
Let A, B, C be three magnitudes, and D, E, F other three, which have the ſame ratio taken two and two, but in a croſs order, viz. as A is to B, ſo is E to F, and as B is to C, ſo is D to E. If A be greater than C, D ſhall be greater than F; and if equal, equal; and if leſs, leſſer.



Becauſe A is greater than C, and B is any other
a. 8. 5. magnitude, A ſhall have to B a greater ratio [a] than C
b. 13. 5. has to B. but as E to F, ſo is A to B; therefor [b] E has to F a greater ratio than C to B. and becauſe B is to C, as D to E, by inverſion, C ſhall be to B, as E to D. and E was ſhewn to have to F a greater ratio than
c. Cor. 13. 5. C to B; therefor E has to F a greater ratio than E to D[c]. but the magnitude to which the ſame has a greater ratio
d. 10. 5. than it has to another, is the leſſer of the two[d]. F therefor is leſſer than D; that is, D is greater than F.

Secondly, Let A be equal to C; D ſhall be equal to F. Becauſe A and
e. 7. 5. C are equal, A ſhall be [e] to B, as C is to B. but A is to B, as E to F; and C is to B, as E to D; wherefor E is
f. 11. 5. to F, as E to D[f]; and therefor D is
g. 9. 5. equal to F[g].



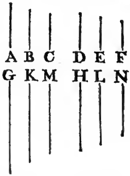


Next, Let A be leſſer than C; D ſhall be leſſer than F. for becauſe C is greater than A, and that, as was ſhewn, C is to B, as E to D, and in like manner B is to A, as F to E; F ſhall be greater than D, by caſe firſt; and therefor D is leſſer than F. Therefor if there be three &c. Q. E. D.

PROP. XXII. THEOR.

IF there be any number of magnitudes, and as many others, which taken two and two in order have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt has to the laſt of the others. N. B. *This is uſually cited by the words "ex "aequali."*

Firſt, Let there be three magnitudes A, B, C, and as many others D, E, F, which two and two have the ſame ratio, that is ſuch that A is to B, as D to E; and as B is to C, ſo is E to F. A ſhall be to C, as D to F.



Take of A and D any equimultiples whatever G and H; and of B and E any equimultiples whatever K and L; and of C and F any whatever M and N. then becauſe A is to B, as D to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E; as G is to K, ſo ſhall[a] H be to L. by the ſame reaſon K ſhall be to M, as L to N. and becauſe there are three magnitudes G, K, M, and other three H, L, N, which two and two have the ſame ratio; if G be greater

a. 4. 5.

than M, H ſhall be greater than N; and if equal, equal; and if leſs, b. 20. 5. leſſer[b]. and G, H are equimultiples of A, D any whatever, and M, c. 5. Def. 5. N are any equimultiples whatever of C, F. as therefor [c] A is to C, ſo is D to F.

Next, Let there be four magnitudes A, B, C, D, and other four E, F, G, H, which two and two have the ſame ratio, viz. as A is to B, ſo is E to F; and as B to C, ſo F to G; and as C to D, ſo G to H. A ſhall be to D, as E to H.

A. B. C. D.
E. F. G. H.

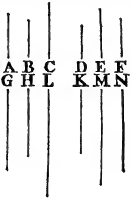
Becauſe A, B, C are three magnitudes, and E, F, G other three, which two and two have the ſame ratio; by the foregoing caſe, A ſhall be to C, as E to G. but C is to D, as G is to H; wherefor again, by the firſt caſe, A ſhall be to D, as E to H. and ſo on, whatever be the number of magnitudes. Therefor if there be any number &c. Q. E. D.

PROP. XXIII. THEOR.

IF there be any number of magnitudes, and as many others, which, taken two and two, in a croſs order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt has to the laſt of the others. N. B. *This is uſually cited by the words " ex aequali in proportione perturbata."*

Firſt, Let there be three magnitudes A, B, C, and other three D, E, F, which taken two and two in a croſs order have the ſame ratio, that is ſuch that A is to B, as E to F; and as B is to C, ſo is D to E. A ſhall be to C, as D to F.

Take of A, B, D any equimultiples whatever G, H, K; and of C, E, C, E, F any equimultiples whatever L, M, N. and becauſe G, H are equimultiples of A, B, and that magnitudes have the ſame ratio which their equimultiples have[a]; as A is to B, ſo is G to H. and by the ſame reaſon, as E is to F, ſo is M to N. but as A is to B, ſo is E to F; as therefor G is to H, ſo is M to N[b]. and becauſe as B is to C, ſo is D to E, and that H, K are equimultiples of B, D, and L, M of C, E; as H is to L, ſo ſhall [c] K be to M. and it has been ſhewn that G is to H, as M to N. becauſe therefor there are three magnitudes G, H, L, and other three K, M, N which have the ſame ratio taken two and two in a croſs order; if G be greater than L, K ſhall be greater than N; and if equal, equal; and if leſs, leſſer. and G, K are any equimultiples whatever of A, D; and L, N any whatever of C, F; as therefor A is to C, ſo is D to F.



a. 15. 5.

b. 11. 5.

c. 4. 5.

Next, Let there be four magnitudes A, B, C, D, and other four E, F, G, H, which, taken two and two in a croſs order, have the ſame ratio, viz. A to B, as G to H; B to C, as F to G; and C to D, as E to F. A ſhall be to D, as E to H.

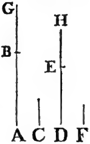
A. B. C. D.
E. F. G. H.

Becauſe A, B, C are three magnitudes, and F, G, H other three, which, taken two and two in a croſs order, have the ſame ratio; by the firſt caſe, A ſhall be to C, as F to H. but C is to D, as E is to F; wherefor again, by the firſt caſe, A ſhall be to D, as E to H. and ſo on, whatever be the number of magnitudes. Therefor if there be any number &c. Q. E. D.

PROP. XXIV. THEOR.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth; and the fifth to the ſecond the ſame ratio which the ſixth has to the fourth; the firſt and fifth together ſhall have to the ſecond, the ſame ratio which the third and ſixth together has to the fourth.

Let AB the firſt have to C the ſecond the ſame ratio, which DE the third has to F the fourth; and let BG the fifth have to C the ſecond the ſame ratio, which EH the ſixth has to F the fourth. AG, the firſt and fifth together, ſhall have to C the ſecond the ſame ratio, which DH, the third and ſixth together, has to F the fourth.



Becauſe BG is to C, as EH to F; by inverſion C ſhall be to BG, as F to EH. and becauſe as AB is to C, ſo is DE to F; and as C to BG, ſo F to a. 22. 5. EH; ex aequali [a] AB ſhall be to BG, as DE to EH. and becauſe theſe magnitudes are proportionals, they ſhall likewiſe be proportionals when taken b. 18. 5. jointly [b]; as therefor AG is to GB, ſo is DH to HE; but as GB to C, ſo is HE to F. Therefor, ex aequali [a], as AG is to C, ſo is DH to F. Wherefor if the firſt &c. Q. E. D.

Cor. 1. If the ſame Hypotheſis be made as in the Propoſition, the exceſs of the firſt and fifth ſhall be to the ſecond, as the exceſs of the third and ſixth to the fourth. the Demonſtration of this is the ſame with that of the Propoſition, if Diviſion be uſed inſtead of Compoſition.

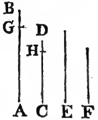
Cor. 2. The Propoſition holds true of two ranks of magnitudes, whatever be their number, of which each of the firſt rank has to a ſecond magnitude the ſame ratio that the correſponding one of the ſecond rank has to a fourth magnitude; as is manifeſt.

PROP. XXV. THEOR.

IF four magnitudes are proportionals, the greateſt and leaſt of them together ſhall be greater than the other two together.

Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greateſt of them, and conſequently F the leaſt[a]. AB together with F ſhall be greater than CD together with E.

Take AG equal to E, and CH equal to F. then becauſe as AB to CD, ſo is E to F, and that AG is equal to E, and CH equal to F; AB ſhall be to CD, as AG to CH. and becauſe AB the whole is to the whole CD, as AG is to CH; likewiſe the remainder GB ſhall be to the remainder HD, as the whole AB is to the whole[b] CD. but AB is greater than CD, therefor[c] GB is greater than HD. and becauſe AG is equal to E, and CH to F; AG and F together ſhall be equal to CH and E together. If therefor unto the unequal magnitudes GB and HD, of which GB is the greater, there be added equal magnitudes, viz. unto GB the two AG and F, and CH and E unto HD; AB and F ſhall be greater than CD and E. Therefor if four magnitudes &c. Q. E. D.



PROP. F.

[a] A. and 14. 5.

[b] 19. 5.

[c] A. 5.

PROP. F. THEOR.

RATIOS which are compounded of the ſame ratios, are the ſame to one another.

Let A be to B, as D to E; and B to C, as E to F. the ratio which is compounded of the ratios of A to B, and B to C, which, by the Definition of compound ratio, is the ratio of A to C, ſhall be the ſame with the ratio of D to F, which, by the ſame Definition, is compounded of the ratios of D to E, and E to F.

A. B. C.
D. E. F.

Becauſe there are three magnitudes A, B, C, and three others D, E, F which taken two and two in order have the ſame ratio; ex aequali,
a. 22. 5. A ſhall be to C, as D to F[a].

Next, Let A be to B, as E to F; and B to C, as D to E; there-
b. 23. 5. for, ex aequali in ratione perturbata[b], A ſhall be to C, as D to F; that is, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the ſame with the ratio of D to F, which is compounded of the ratios of D to E, and E to F. and in like manner the Propoſition ſhall be demonſtrated whatever be the number of ratios in either caſe.

A. B. C.
D. E. F.

PROP. G. THEOR.

IF ſeveral ratios be the ſame to ſeveral ratios, each to each; the ratio which is compounded of ratios which are the ſame to the firſt ratios, each to each, ſhall be the ſame to the ratio which is compounded of ratios which are the ſame to the other ratios, each to each.

Let A be to B, as E to F; and C to D, as G to H. and let A be to B, as K to L; and C to D, as L to M. then the ratio of K to M, by the Definition of compound ratio, is compounded of the ratios of K to L, and L to M, which are the ſame with the ratios of A to B, and C to D. and as E to F, ſo let N be to O; and as G to H, ſo let O be to P; then the ratio of N to P is compounded of the ratios of N to O, and O to P, which are the ſame with the ratios of E to F, and G to H. and it is to be ſhewn that the ratio of K to M, is the ſame with the ratio of N to P, or that K is to M, as N to P.

A. B. C. D.	K. L. M.
E. F. G. H.	N. O. P.

Becauſe K is to L, as (A to B, that is, as E to F, that is as) N to O; and as L to M, ſo is (C to D, and ſo is G to H, and ſo is) O to P. ex aequali[a], K ſhall be to M, as N to P. Therefor if ſeveral ratios &c. Q. E. D. a. 22. 5.

PROP. H. THEOR.

IF a ratio which is compounded of ſeveral ratios be the ſame to a ratio which is compounded of ſeveral ratios, and if one of theſe firſt ratios, or the ratio which is compounded of ſeveral of them, be the ſame to one of the other ratios, or to the ratio which is compounded of ſeveral of them; then the remaining ratio of theſe firſt, or, if there be more than one, the ratio compounded of the remaining ratios, ſhall be the ſame to the remaining ratio of the other ratios, or, if there be more than one, to the ratio compounded of theſe remaining ratios.

Let the firſt ratios be thoſe of A to B, B to C, C to D, D to E, and E to F; and let the other ratios be thoſe of G to H, H to K, K to L, and L to M. alſo let the ratio of A to F, which is compounded [a] of the firſt ratios be the ſame with the ratio of G to M, which is compounded of the other ratios. and beſides, let the ratio of A to D, which is compounded of the ratios of A to B, B to C, C to D, be the ſame with the ratio of G to K, which is compounded of the ratios of G to H, and H to K. then ſhall the ratio compounded of the remaining firſt ratios, to wit, of the ratios of D to E, and E to F, which compounded ratio is the ratio of D to F, be the ſame with the ratio of K to M, which is compounded of the remaining ratios of K to L, and L to M of the other ratios.

A. B. C. D. E. F.
G. H. K. L. M.

a. Definition of compounded ratio.

Becauſe, by the Hypotheſis, A is to D, as G to K, by inverſion [b], D ſhall be to A, as K to G; and as A is to F, ſo is G to M; therefor [c], ex aequali, D ſhall be to F, as K to M. If therefor a ratio which is &c. Q. E. D.

b. B. 5.

c. 22. 5.

PROP. K. THEOR.

IF there be any number of ratios, and any number of other ratios ſuch, that the ratio which is compounded of ratios which are the ſame to the firſt ratios, each to each, is the ſame to the ratio which is compounded of ratios which are the ſame, each to each, to the other ratios; and if one of the firſt ratios, or the ratio which is compounded of ratios which are the ſame to ſeveral of the firſt ratios, each to each, be the ſame to one of the other ratios, or to the ratio which is compounded of ratios which are the ſame, each to each, to ſeveral of the other ratios: then the remaining ratio of the firſt, or, if there be more than one, the ratio which is compounded of ratios which are the ſame, each to each, to the remaining ratios of the firſt, ſhall be the ſame to the remaining ratio of the other ratios, or, if there be more than one, to the ratio which is compounded of ratios which are the ſame to the remaining other ratios, each to each.

Let the ratios of A to B, C to D, E to F be the firſt ratios; and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios. and let A be to B, as S to T; and C to D, as T to V; and E to F, as V to X. therefor, by the Definition of compound ratio, the ratio of S to X is compounded of the ratios of S to T, T to V, and V to X, which are the ſame to the ratios of A to B, C to D, E to F, each to each. alſo as G to H, ſo let Y be to Z; and K to L, as Z to a; M to N, as a to b; O to P, as b to c; and Q to R, as c to d. therefor, by the ſame Definition, the ratio of Y to d is compounded of the ratios of Y to Z, Z to a, a to b, b to c, and c to d, which are the ſame, each to each, to the ratios of G to H, K to L, M to N, O to P, and Q to R. therefor, by the Hypotheſis, S is

h, k, l.
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d.
e, f, g. m, n, o, p.

to X, as Y to d. alſo let the ratio of A to B, that is, the ratio of S to T, which is one of the firſt ratios, be the ſame to the ratio of e to g, which is compounded of the ratios of e to f, and f to g, which, by the Hypotheſis, are the ſame to the ratios of G to H, and K to L, two of the other ratios; and let the ratio of h to l be that which is compounded of the ratios of h to k, and k to l, which are the ſame to the remaining firſt ratios, viz. of C to D, and E to F; alſo let the ratio of m to p be that which is compounded of the ratios of m to n, n to o, and o to p, which are the ſame, each to each, to the remaining other ratios, viz. of M to N, O to P, and Q to R. then the ratio of h to l ſhall be the ſame to the ratio of m to p, or h ſhall be to l, as m to p.

h, k, l.
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d.
e, f, g. m, n, o, p.

Becauſe e is to f, as (G to H, that is as) Y to Z; and f is to g, as (K to L, that is as) Z to a; therefor, ex aequali, e is to g, as Y to a. and, by the Hypotheſis, A is to B, that is S to T, as e to g; wherefor S is to T, as Y to a, and, by inverſion, T is to S, as a to Y; and S is to X, as Y to d; therefor, ex aequali T is to X, as a to d. alſo becauſe h is to k, as (C to D, that is as) T to V; and k is to l, as (E to F, that is as) V to X; therefor, ex aequali, h is to l, as T to X. in the like manner it ſhall be demonſtrated that m is to p, as a to d. and it was ſhewn that T is to X, as a to d. therefor [a] h is to l, as m to p. Q. E. D.

a. 11. 5.

The Propoſitions G and K are uſually, for the ſake of brevity, expreſſed in the very ſame terms with Propoſitions F and H. and therefor it was proper to ſhew the true meaning of them when they are ſo expreſſed; eſpecially ſince they are moſt frequently made uſe of by Geometers.

THE
ELEMENTS
OF
EUCLID.

BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are thoſe which have their ſeveral angles equal, each to each, and the ſides about the equal angles proportionals.

II.

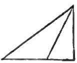
" Reciprocal figures, viz. triangles and parallelograms, are ſuch as have " their ſides about two of their angles proportionals in ſuch manner, that a ſide of the firſt figure is to a ſide of the other as " the remaining ſide of this other is to the remaining ſide of the " firſt."

III.

A ſtraight line is ſaid to be cut in extreme and mean ratio, when the whole is to the greater ſegment, as the greater ſegment is to the leſs.

IV.

The altitude of any figure is the ſtraight line drawn from its vertex perpendicular to the baſe.



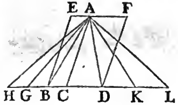
PROP. I. THEOR.

Triangles and parallelograms of the ſame altitude are one to another as their baſes.

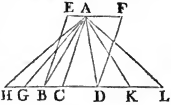
Let the triangles ABC, ACD, and the parallelograms EC, CF have the ſame altitude, viz. the perpendicular drawn from the point A to BD. as the baſe BC is to the baſe CD, ſo ſhall the triangle ABC be to the triangle ACD, and the parallelogram EC to the parallelogram CF.

a. 38. 1.

Produce BD both ways to the points H, L, and take any number of ſtraight lines BG, GH, each equal to the baſe BC; and DK, KL, any number of them, each equal to the baſe CD; and join AG, AH, AK, AL. then becauſe CB, BG, GH are all equal, the triangles AHG, AGB, ABC ſhall be all equal[a]. therefor whatever multiple the baſe HC is of the baſe BC, the ſame multiple is the triangle AHC of the triangle ABC. by the ſame reaſon whatever multiple the baſe LC is of the baſe CD, the ſame multiple is the triangle ALC of the triangle ADC. and if the baſe HC be equal to the baſe CL, the triangle AHC ſhall alſo be equal to the triangle ALC[a]; and if the baſe HC be greater than the baſe CL, likewiſe the triangle AHC ſhall be greater than the triangle ALC; and if leſs, leſſer. therefor ſince there are four magnitudes, viz. the two baſes



baſes BC, CD, and the two triangles ABC, ACD; and of the baſe BC and the triangle ABC the firſt and third, any equimultiples whatever have been taken, viz. the baſe HC and triangle AHC; and of the baſe CD and triangle ACD the ſecond and fourth have been taken any equimultiples whatever, viz. the baſe CL and triangle ALC; and it has been ſhewn that if the baſe HC be greater than the baſe CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; b. 5. Def. 5. and if leſs, leſſer. Therefor[b] as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD.



And becauſe the parallelogram c. 41. 1. CE is double of the triangle ABC[c], and the parallelogram CF double of the triangle ACD, and that magd. 15. 5. nitudes have the ſame ratio which their equimultiples have[d]; as the triangle ABC is to the triangle ACD, ſo ſhall the parallelogram EC be to the parallelogram CF. and becauſe it has been ſhewn that as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD; and as the triangle ABC to the triangle ACD, ſo is the parallelogram EC to the parallelogram CF; therefor as the baſe BC is to e. 11. 5. the baſe CD, ſo ſhall [e] the parallelogram EC be to the parallelogram CF. Wherefor triangles &c. Q. E. D.

Cor. From this it is plain that triangles and parallelograms that have equal altitudes, are one to another as their baſes.

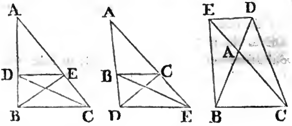
Let the figures be placed ſo as to have their baſes in the ſame ſtraight line; and having drawn perpendiculars from the vertices of the triangles to the baſes, the ſtraight line which joins the vertices ſhall f. 33. 1. be parallel to that in which their baſes are[f], becauſe the perpendiculars are both equal and parallel to one another. then, if the ſame conſtruction be made as in the Propoſition, the Demonſtration ſhall be the ſame.

PROP. II. THEOR.

IF a ſtraight line be drawn parallel to one of the ſides of a triangle, it ſhall cut the other ſides, or theſe ſides produced, proportionally. and if the ſides, or the ſides produced be cut proportionally, the ſtraight line which joins the points of ſection ſhall be parallel to the remaining ſide of the triangle.

Let DE be drawn parallel to BC one of the ſides of the triangle ABC. BD ſhall be to DA, as CE to EA.

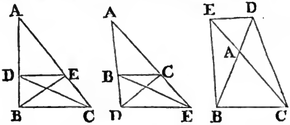
Join BE, CD; then the triangle BDE is equal to the triangle CDE[a], becauſe they are on the ſame baſe DE, and between the a. 37. 1. ſame parallels DE, BC. ADE is another triangle, and equal magni-



tudes have to the ſame, the ſame ratio[b]; therefor as the triangle BDE b. 7. 5. to the triangle ADE, ſo is the triangle CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, ſo is [c]BD to DA, be- c. 1. 6. cauſe having the ſame altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their baſes. and by the

ſame reaſon, as the triangle CDE to the triangle ADE, ſo is CE to
d. 11. 5. EA. Therefor as BD to DA; ſo is CE to EA[d].

Next, Let the ſides AB, AC of the triangle ABC, or theſe ſides



produced, be cut proportionally in the points D, E, that is, ſo that BD be to DA, as CE to EA; and join DE. DE ſhall be parallel to BC.

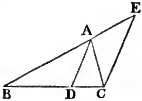
The ſame conſtruction being made, becauſe as BD to DA, ſo is CE to EA; and as BD to DA, ſo is the triangle BDE to the tri-
c. 1. 6. angle ADE[c]; and as CE to EA, ſo is the triangle CDE to the triangle ADE; the triangle BDE ſhall be to the triangle ADE, as the triangle CDE to the triangle ADE. Wherefor the triangles BDE,
e. 9. 5. CDE have the ſame ratio to the triangle ADE; and therefor [e] the triangle BDE is equal to the triangle CDE. and they are on the ſame baſe DE; but equal triangles on the ſame baſe are between the ſame
f. 39. 1. parallels[f]; therefor DE is parallel to BC. Wherefor if a ſtraight line &c. Q. E. D.

PROP. III. THEOR.

If the angle of a triangle be divided into two equal angles, by a ſtraight line which alſo cuts the baſe; the ſegments of the baſe ſhall have the ſame ratio which the other ſides of the triangle have to one another. and if the ſegments of the baſe have the ſame ratio which the other ſides of the triangle have to one another, the ſtraight line drawn from the vertex to the point of ſection ſhall divide the angle of the triangle into two equal angles.

Let the angle BAC of any triangle ABC be divided into two equal angles by the ſtraight line AD. BD ſhall be to DC, as BA to AC.

Thro' the point C draw CE parallel [a] to DA, and let BA produced a. 31. 1.
meet CE in E. Becauſe the ſtraight line AC meets the parallels
AD, EC, the angle ACE ſhall be equal to the alternate angle CAD[b]. b. 29. 1.
but CAD, by the Hypotheſis, is equal to the angle BAD; wherefor
BAD is equal to the angle ACE. Again, becauſe the ſtraight line
BAE meets the parallels AD, EC,
the outward angle BAD is equal to the
inward and oppoſite angle AEC. but
the angle ACE has been proved equal to
the angle BAD; therefor alſo ACE is
equal to the angle AEC; and the ſide
AE ſhall be equal to the ſide [c] AC. c. 6. 1.
and becauſe AD is drawn parallel to one of the ſides of the triangle
BCE, viz. to EC, BD ſhall be to DC, as BA to AE[d]; but AE is d. 2. 6.
equal to AC; therefor as BD to DC, ſo is BA to AC[e]. e. 7. 5.



Let now BD be to DC, as BA to AC, and join AD; the angle BAC ſhall be divided into two equal angles by the ſtraight line AD.

The ſame conſtruction being made, becauſe as BD to DC, ſo is
BA to AC; and as BD to DC, ſo is BA to AE[d]; BA ſhall be to
AC, as BA to AE[f]. therefor AC is equal to AE[g], and the angle f. 11. 5.
g. 9. 5.

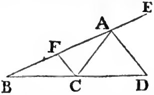
AEC ſhall be equal to the angle ACE[h]. but the angle AEC is equal
h. 5. 1. to the outward and oppoſite angle BAD; and the angle ACE is equal
b. 29. 1. to the alternate angle CAD[b]. wherefor alſo the angle BAD is equal to the angle CAD. therefor the angle BAC is cut into two equal angles by the ſtraight line AD. therefor if the angle &c. Q. E. D.

PROP. A. THEOR.

IF the outward angle of a triangle made by producing one of its ſides, be divided into two equal angles, by a ſtraight line which alſo cuts the baſe produced; the ſegments between the dividing line and the extremities of the baſe ſhall have the ſame ratio which the other ſides of the triangle have to one another. and if the ſegments of the baſe produced have the ſame ratio which the other ſides of the triangle have, the ſtraight line drawn from the vertex to the point of ſection ſhall divide the outward angle of the triangle into two equal angles.

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the ſtraight line AD which meets the baſe produced in D. BD ſhall be to DC, as BA to AC.

a. 31. 1. Through C draw CF parallel to AD[a]; and becauſe the ſtraight line AC meets the parallels AD, FC, the angle ACF ſhall be equal to the
b. 29. 1. alternate angle CAD[b]. but CAD is e-
c. Hyp. qual to the angle DAE[c]; therefor alſo DAE ſhall be equal to the angle ACF. Again, becauſe the ſtraight line FAE meets the parallels AD, FC, the outward angle DAE is equal to the



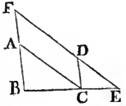
inward and oppoſite angle CFA. but the angle ACF has been proved equal to the angle DAE; therefor alſo the angle ACF is equal to the angle CFA, and the ſide AF ſhall be equal to the ſide AC[d]. and d. 6. 1. becauſe AD is parallel to FC a ſide of the triangle BCF, BD ſhall be to DC, as BA to AF[e]; but AF is equal to AC; as therefor BD is e. 2. 6. to DC, ſo is BA to AC.

Let now BD be to DC, as BA to AC, and join AD; the angle CAD ſhall be equal to the angle DAE.

The ſame conſtruction being made, becauſe BD is to DC, as BA to AC; and that BD is alſo to DC, as BA to AF[e]; BA ſhall be to AC, as BA to AF[f]. therefor AC is equal to AF[g], and the angle f. 11. 5. AFC ſhall be equal [h] to the angle ACF. but the angle AFC is equal g. 9. 5. h. 5. 1. to the outward angle EAD, and the angle ACF to the alternate angle CAD; therefor alſo EAD is equal to the angle CAD. Wherefor if the outward &c. Q. E. D.

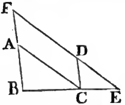
PROP. IV. THEOR.

THE ſides about the equal angles of equiangular triangles are proportionals; and thoſe which are oppoſite to the equal angles are homologous ſides, i. e. are the antecedents or conſequents of the ratios.



Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and conſequently [a] the angle BAC equal to a. 32. 1. the angle CDE. The ſides about the equal angles of the triangles ABC, DCE ſhall be proportionals; and theſe ſhall be the homologous ſides which are oppoſite to the equal angles.

Let the triangle DCE be placed ſo that its ſide CE may be contiguous
b. 22. 1. to BC, and in the ſame ſtraight line with it[b]. and becauſe the
c. 17. 1. angles ABC, ACB are together leſs than two right angles[c]; ABC
and DEC, which is equal to ACB, ſhall alſo be leſs than two right
d. 12. Ax. 1. angles. wherefor BA, ED produced ſhall meet[d]; let them be produced
and meet in the point F. and becauſe the angle ABC is equal
e. 28. 1. to the angle DCE, BF ſhall be parallel [e] to CD. Again, becauſe the
angle ACB is equal to the angle DEC, AC
ſhall be parallel to FE[e]. therefor FACD is
a parallelogram; and AF ſhall be equal to
f. 34. 1. CD, and AC to FD[f]. and becauſe AC is
parallel to FE one of the ſides of the triangle
FBE, BA ſhall be to AF, as BC to
g. 2. 6. CE[g]. but AF is equal to CD, therefor [h] as BA to CD, ſo is BC to
h. 7. 5. CE; and alternately, as AB to BC, ſo DC to CE. Again, becauſe CD
is parallel to BF, as BC to CE, ſo is FD to DE[g]; but FD is equal
to AC; therefor as BC to CE, ſo is AC to DE. and alternately,
as BC to CA, ſo CE to ED. therefor becauſe it has been proved
that AB is to BC, as DC to CE; and as BC to CA, ſo CE to ED, ex
i. 22. 5. aequali[i], BA ſhall be to AC, as CD to DE. Therefor if the ſides &c.
Q. E. D.

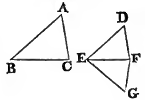


PROP. V. THEOR.

IF the ſides of two triangles, about each of their angles, be proportionals, the triangles ſhall be equiangular. and the equal angles ſhall be thoſe which are oppoſite to the homologous ſides.

Let the triangles ABC, DEF have their ſides proportionals, ſo that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and conſequently, ex aequali, BA to AC, as ED to DF. the triangle ABC ſhall be equiangular to the triangle DEF, and the angles oppoſite to their homologous ſides ſhall be equal, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and alſo BAC to EDF.

At the points E, F in the ſtraight line EF make [a] the angle FEG a. 23. 1. equal to the angle ABC, and the angle EFG equal to BCA; wherefor the remaining angle BAC ſhall be equal to the remaining angle EGF[b], b. 32. 1. and the triangle ABC ſhall be equiangular to the triangle DEF; and they ſhall have their ſides oppoſite to the equal angles proportionals[c]. c. 4. 6. wherefor as AB to BC, ſo is GE to EF; but as AB to BC, ſo is DE to EF; therefor as DE to EF, ſo [d] GE to EF. d. 11. 5. therefor DE and GE have the ſame ratio to EF, and ſhall be equal[e]. e. 9. 5. by the ſame reaſon DF is equal to FG. and becauſe, in the triangles DEF, GEF, DE is equal to EG, and EF common, the two ſides DE, EF are equal to the two GE, EF, and the baſe DF is equal to the baſe GF; therefor the angle DEF is equal [f] to the angle GEF, f. 8. 1. and the other angles to the other angles which are ſubtended by the equal ſides[g]. Wherefor the angle DFE is equal to the angle GFE, g. 4. 1. and EDF to EGF. and becauſe the angle DEF is equal to the angle GEF, and GEF to the angle ABC; the angle ABC ſhall be equal to the angle DEF. by the ſame reaſon, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D. Therefor the triangle ABC is equiangular to the triangle DEF. Wherefor if the ſides &c. Q. E. D.

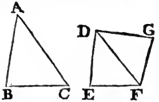


PROP. VI.

PROP. VI. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about theſe equal angles proportionals, the triangles ſhall be equiangular, and ſhall have thoſe angles equal which are oppoſite to the homologous ſides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the ſides about theſe angles proportionals; that is, BA to AC, as ED to DF. The triangles ABC, DEF ſhall be equiangular, and ſhall have the angle ABC equal to the angle DEF, and ACB to DFE.



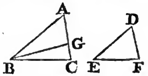
a. 23. 1. At the points D, F, in the ſtraight line DF, make [a] the angle
FDG equal to either of the angles BAC, EDF; and the angle DFG
equal to the angle ACB. wherefor
the remaining angle at B ſhall be
b. 32. 1. equal to the remaining one at G [b],
and the triangle ABC be equiangular
to the triangle DGF; and therefor
c. 4. 6. as BA to AC, ſo is [c] GD to DF.
but, by the Hypotheſis, as BA to
d. 11. 5. AC, ſo is ED to DF; as therefor ED to DF, ſo is [d] GD to DF;
e. 9. 5. wherefor ED is equal [e] to DG; and DF is common to the two triangles
EDF, GDF. therefor the two ſides ED, DF are equal to
the two ſides GD, DF; and the angle EDF is equal to the angle
f. 4. 1. GDF, wherefor the baſe EF is equal to the baſe FG [f], and the triangle
EDF to the triangle GDF, and the remaining angles to the
remaining angles, each to each, which are ſubtended by the equal
ſides.

ſides. therefor the angle DFG is equal to the angle DFE, and the angle at G to the angle at E. but the angle DFG is equal to the angle ACB; therefor the angle ACB is equal to the angle DFE. and the angle BAC is equal to the angle EDF[g]; wherefor alſo the remaining angle at B is equal to the remaining angle at E. Therefor the triangle ABC is equiangular to the triangle DEF. Wherefor if two triangles &c. Q. E. D.

PROP. VII. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about two other angles proportionals; then if each of the remaining angles be either leſs, or not leſs than a right angle; or if one of them be a right angle: the triangles ſhall be equiangular, and ſhall have thoſe angles equal about which the ſides are proportionals.

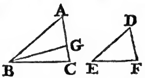
Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the ſides about two other angles ABC, DEF proportionals, ſo that AB is to BC, as DE to EF; and, in the firſt caſe, let each of the remaining angles at C, F be leſs than a right angle. The triangle ABC ſhall be equiangular to the triangle DEF, viz. the angle ABC ſhall be equal to the angle DEF, and the remaining angle at C to the remaining angle at F.



For if the angles ABC, DEF be not equal, one of them ſhall be greater than the other; let ABC be the greater, and at the point B in the

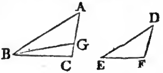
[g] Hyp.

the ſtraight line AB make the angle ABG equal to the angle [a] DEF.
a. 23. 1. and becauſe the angle at A is equal to the angle at D, and the angle
ABG to the angle DEF; the remaining
b. 32. 1. angle AGB ſhall be equal [b] to the remaining angle DFE. therefor the triangle ABG is equiangular to the tri-
c. 4. 6. angle DEF; wherefor [c] as AB is to



BG, ſo is DE to EF; but as DE to EF, ſo, by Hypotheſis, is AB to
d. 11. 5. BC; therefor as AB to BC, ſo is AB to BG [d]; and becauſe AB has
e. 9. 5. the ſame ratio to each of BC, BG; BC ſhall be equal [e] to BG, and
f. 5. 1. therefor the angle BGC is equal to the angle BCG [f]. but the angle BCG is, by Hypotheſis, leſs than a right angle; therefor alſo the angle BGC is leſs than a right angle, and the adjacent angle AGB muſt be
g. 13. 1. greater than a right angle [g]. But it was proved that the angle AGB is equal to the angle at F; therefor the angle at F is greater than a right angle. but, by the Hypotheſis, it is leſs than a right angle; which is abſurd. Therefor the angles ABC, DEF are not unequal, that is, they are equal. and the angle at A is equal to the angle at D; wherefor the remaining angle at C is equal to the remaining angle at F. therefor the triangle ABC is equiangular to the triangle DEF.

Next, let each of the angles at C, F be not leſs than a right angle. the triangle ABC ſhall likewiſe in this caſe be equiangular to the triangle DEF.

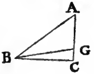


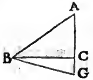
The ſame conſtruction being made, in like manner it ſhall be proved that BC is equal to BG, and the angle at C equal to the angle BGC. but the angle at C is not leſs than a right angle; therefor the angle BGC is not leſs than a right angle. wherefor two angles of the triangle

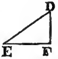
angle BGC are together not leſs than two right angles; which is impoſſible[h]; and therefor the triangle ABC ſhall be proved to be equiangular to the triangle DEF, as in the firſt caſe.

h. 17. 1.

Laſtly, Let one of the angles at C, F, viz. the angle at C be a right angle; likewiſe in this caſe the triangle ABC ſhall be equiangular to the triangle DEF.







For if they be not equiangular, make at the point B of the ſtraight line AB the angle ABG equal to the angle DEF; and it ſhall be proved, as in the firſt caſe, that BG is equal to BC. but the angle BCG is a right angle, therefor[f] the angle BGC is alſo a right angle; whence two of the angles of the triangle BGC are together not leſs than two right angles; which is impoſſible[h]. therefor the triangle ABC ſhall be equiangular to the triangle DEF. Wherefor if two triangles &c. Q. E. D.

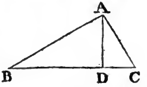
f. 5. 1.

PROP. VIII. THEOR.

IN a right angled triangle, if a perpendicular be drawn from the right angle to the baſe; the triangles on each ſide of it ſhall be ſimilar to the whole triangle, and to one another.

Let ABC be a right angled triangle having the right angle BAC; and from the point A let AD be drawn perpendicular to the baſe BC. the triangles ABD, ADC ſhall be ſimilar to the whole triangle ABC, and to one another.

Becauſe the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB ſhall be equal to the remaining angle BAD[a]. therefor the triangle ABC is equiangular to the triangle ABD, and the ſides about their equal angles ſhall be proportionals[b], wherefor they are ſimilar to one another[c]. in the like manner it ſhall be demonſtrated that the triangle ADC is ſimilar to the triangle ABC.



a. 32. 1.
b. 4. 6.
c. 1. Def. 6.

Alſo the triangles ABD, ADC ſhall be ſimilar to one another.

Becauſe the right angle BDA is equal to the right angle ADC, and alſo the angle BAD to the angle at C, as has been proved; the remaining angle at B ſhall be equal to the remaining angle DAC. therefor the triangle ABD is equiangular and ſimilar[c] to the triangle ADC. Therefor in a right angled &c. Q. E. D.

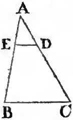
COR. From this it is manifeſt that the perpendicular drawn from the right angle of a right angled triangle to the baſe, is a mean proportional between the ſegments of the baſe: and alſo that each of the ſides is a mean proportional between the baſe, and the ſegment of it adjacent to that ſide. becauſe in the triangles BDA, ADC, BD is to DA, as DA to DC[b]; and in the triangles ABC, DBA, BC is to BA, as BA to BD[b]; and in the triangles ABC, ACD, BC is to CA, as CA to CD[b].

PROP. IX. PROB.

FROM a given ſtraight line to cut off any part required.

Let AB be the given ſtraight line; it is required to cut off any part from it.

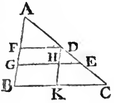
From the point A draw a ſtraight line AC making any angle with AB; and in AC take any point D, and take AC which is the ſame multiple of AD that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it. AE ſhall be the ſame part of AB that AD is of AC; that is, AE ſhall be the part required to be cut off.



Becauſe ED is parallel to one of the ſides of the triangle ABC, viz. to BC, as CD is to DA, ſo is [a] a. 2. 6.
BE to EA; and, by compoſition [b], CA is to AD, as BA to AE. but b. 18. 5.
CA is a multiple of AD, therefor [c] BA is the ſame multiple of AE. c. D. 5.
whatever part therefor AD is of AC, AE is the ſame part of AB. wherefor from the ſtraight line AB the part required is cut off. Which was to be done.

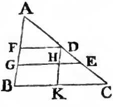
PROP. X. PROB.

TO divide a given ſtraight line ſimilarly to a given divided ſtraight line, i. e. into parts that ſhall have the ſame ratios to one another which the parts of the divided given ſtraight line have.



Let AB be the ſtraight line given to be divided, and AC the divided line; it is required to divide AB ſimilarly to AC.

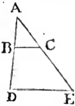
Let AC be divided in the points D, E; and let AB, AC be placed ſo as to contain any angle, and join BC, and through the points D, E draw [a] DF, EG parallels to it; and through D draw DHK parallel to AB. a. 31. 1. therefor each of the figures FH, HB is a parallelogram; b. 34. 1. wherefor DH is equal [b] to FG, and HK to GB. and becauſe HE is parallel to KC one of the ſides of the triangle c. 2. 6. DKC, as CE to ED, ſo ſhall [c] KH be to HD. but KH is equal to BG, and HD to GF; therefor as CE to ED, ſo is BG to GF. again, becauſe FD is parallel to EG one of the ſides of the triangle AGE, as ED to DA, ſo ſhall GF be to FA. but it has been proved that CE is to ED, as BG to GF; as therefor CE to ED, ſo is BG to GF; and as ED to DA, ſo GF to FA. therefor the given ſtraight line AB is divided ſimilarly to AC. Which was to be done.



PROP. XI. PROB.

TO find a third proportional to two given ſtraight lines.

Let AB, AC be the two given ſtraight lines, and let them be placed ſo as to contain any angle; it is required to find a third proportional to AB, AC.



Produce AB, AC to the points D, E; and make BD equal to AC, and having joined BC, through D a. 31. 1. draw DE parallel to it [a].

Becauſe BC is parallel to DE a ſide of the triangle b. 2. 6. ADE, AB ſhall [b] be to BD, as AC to CE. but BD is equal to AC, as therefor AB to AC, ſo is AC to CE. Wherefor to the two given ſtraight lines AB, AC a third proportional CE is found. Which was to be done.

PROP. XII.

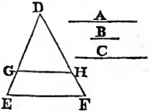
PROP. XII. PROB.

TO find a fourth proportional to three given ſtraight lines.

Let A, B, C be the three given ſtraight lines; it is required to find a fourth proportional to A, B, C.

Take two ſtraight lines DE, DF containing any angle EDF; and upon theſe make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel[a] to it through the point E. and becauſe GH is parallel to EF one of the ſides of the triangle DEF, DG ſhall be to GE, as DH to HF[b]. but DG is equal to A, GE to B, and DH to C; therefor as A is to B, ſo C is to HF. Wherefor to the three given ſtraight lines A, B, C a fourth proportional HF is found. Which was to be done.

a. 31. 1.
b. 2. 6.



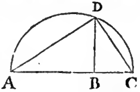
PROP. XIII. PROB.

TO find a mean proportional between two given ſtraight lines.

Let AB, BC be the two given ſtraight lines; it is required to find a mean proportional between them.

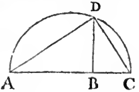
Place AB, BC in a ſtraight line, and upon AC deſcribe the ſemicircle ADC, and from the point B draw[a] BD at right angles to AC, and join AD, DC.

a. 11. 1.



Becauſe the angle ADC in a ſemicircle is a right angle[b], and becauſe in the right angled triangle ADC, DB is drawn from the right angle perpendicular to the baſe, DB ſhall be a mean proportional between AB, BC the ſegments of the baſe[c]. therefor between the two given ſtraight lines AB, BC, a mean proportional DB is found. Which was to be done.

b. 31. 3.
c. Cor. 8. 6.

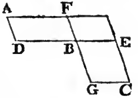


PROP. XIV. THEOR.

EQUAL parallelograms which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional: and parallelograms that have one angle of the one equal to one angle of the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms which have the angles at B equal, and let the ſides DB, BE be placed in the ſame ſtraight line; wherefor alſo FB, BG ſhall be in one ſtraight line[a]. the ſides of the parallelograms AB, BC about the equal angles, ſhall be reciprocally proportional; that is, DB ſhall be to BE, as GB to BF.

a. 14. 1.



Compleat the parallelogram FE; and becauſe the parallelogram AB is equal to BC, and FE is another parallelogram, AB ſhall be to FE, as BC to FE[b]. but as AB to FE, ſo is the baſe DB to BE[c]; and as BC to FE, ſo is the baſe GB to BF; therefor

b. 7. 5.
c. 1. 6.

therefor as DB to BE, ſo is GB to BF[d]. Wherefor the ſides of the parallelograms AB, BC about their equal angles are reciprocally proportional. d. 11. 5.

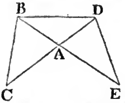
But let the ſides about the equal angles be reciprocally proportional, viz. as DB to BE, ſo GB to BF; the parallelogram AB ſhall be equal to the parallelogram BC.

Becauſe as DB is to BE, ſo GB to BF; and as DB to BE, ſo is the parallelogram AB to the parallelogram FE; and as GB to BF, ſo is parallelogram BC to parallelogram FE; therefor as AB to FE, ſo BC to FE[d]. wherefor the parallelogram AB is equal[e] to the parallelogram BC. Therefor equal parallelograms &c. Q. E. D. e. 9. 5.

PROP. XV. THEOR.

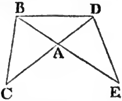
EQUAL triangles which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional: and triangles which have one angle in the one equal to one angle in the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles which have the angle BAC equal to the angle DAE; the ſides about the equal angles of the triangles ſhall be reciprocally proportional; that is, CA ſhall be to AD, as EA to AB.



Let the triangles be placed ſo that their ſides CA, AD be in one ſtraight line; wherefor alſo EA and AB ſhall be in one ſtraight line[a]; and join BD. Becauſe the triangle ABC is equal to the triangle ADE, and ABD a. 14. 1.

is another triangle; therefor as the triangle CAB is to triangle BAD
b. 7. 5. ſo is triangle EAD to triangle DAB[b]. but as triangle CAB to triangle
c. 1. 6. BAD, ſo is the baſe CA to AD[c]; and as triangle EAD to triangle
DAB, ſo is the baſe EA to AB[c]; as therefor CA to AD, ſo is EA
d. 11. 5. to AB[d]. wherefor the ſides of the triangles ABC, ADE about the equal angles are reciprocally proportional.



But let the ſides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB; the triangle ABC ſhall be equal to the triangle ADE.

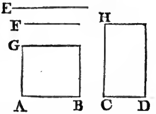
Having joined BD as before, becauſe as CA to AD, ſo is EA to AB; and as CA to AD, ſo is triangle BAC to triangle BAD[c]; and as EA to AB, ſo triangle EAD to triangle BAD[c]; therefor [d] as triangle BAC to triangle BAD, ſo is triangle EAD to triangle BAD; that is, the triangles BAC, EAD have the ſame ratio to the triangle
e. 9. 5. BAD. wherefor the triangle ABC is equal [e] to the triangle ADE.
Therefor equal triangles &c. Q. E. D.

PROP. XVI. THEOR.

IF four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four ſtraight lines ſhall be proportionals.

Let the four ſtraight lines AB, CD, E, F be proportionals, viz. as AB to CD, ſo E to F; the rectangle contained by AB, F ſhall be equal to the rectangle contained by CD, E.

From the points A, C draw [a] AG, CH at right angles to AB, CD; and make AG equal to F, and CH equal to E, and compleat the parallelograms BG, DH. becauſe as AB to CD, ſo is E to F; and that E is equal to CH, and F to AG; AB ſhall [b] be to CD, as CH to AG. therefor the ſides of the parallelograms BG, DH about the equal angles are reciprocally proportional; but parallelograms which have their ſides about equal angles reciprocally proportional, are equal to one another [c]; therefor the parallelogram BG is equal to the parallelogram DH. and the parallelogram BG is contained by the ſtraight lines AB, F, becauſe AG is equal to F; and the parallelogram DH is contained by CD and E, becauſe CH is equal to E. therefor the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD and E.



And if the rectangle contained by the ſtraight lines AB, F be equal to that which is contained by CD, E; theſe four lines ſhall be proportionals, viz. AB ſhall be to CD, as E to F.

The ſame conſtruction being made, becauſe the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD, E, and that the rectangle BG is contained by AB, F, becauſe AG is equal to F; and the rectangle DH by CD, E, becauſe CH is equal to E; therefor the parallelogram BG is equal to the parallelogram DH; and they are equiangular. but the ſides about the equal angles of equal parallelograms are reciprocally proportional [c]. wherefor as AB to CD, ſo is CH to AG; and CH is equal to E, and AG to F. as therefor AB is to CD, ſo E to F. Wherefor if four &c. Q. E. D.

[a] 11. 1.

[b] 7. 5.

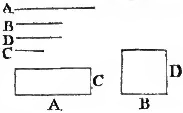
[c] 14. 6.

PROP. XVII. THEOR.

IF three ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the ſquare of the mean: and if the rectangle contained by the extremes be equal to the ſquare of the mean, the three ſtraight lines ſhall be proportionals.

Let the three ſtraight lines A, B, C be proportionals, viz. as A to B, ſo B to C; the rectangle contained by A, C ſhall be equal to the ſquare of B.

Take D equal to B; and becauſe as A to B, ſo B to C, and that a. 7. 5. B is equal to D; A ſhall be [a] to B, as D to C. but if four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the b. 16. 6. means [b]. therefor the rectangle contained by A, C is equal to that contained by B, D. but the rectangle contained by B, D is the ſquare of B; becauſe B is equal to D. therefor the rectangle contained by A, C is equal to the ſquare of B.



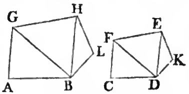
And if the rectangle contained by A, C be equal to the ſquare of B; A ſhall be to B, as B to C.

The ſame conſtruction being made, becauſe the rectangle contained by A, C is equal to the ſquare of B, and the ſquare of B is equal to the rectangle contained by B, D, becauſe B is equal to D; the rectangle contained by A, C ſhall be equal to that contained by B, D. but if the rectangle contained by the extremes be equal to that contained by the means, the four ſtraight lines ſhall be proportionals[b]. therefor A is to B, as D to C; but B is equal to D; wherefor as A to B, ſo B to C. Therefor if three ſtraight lines &c. Q. E. D. b. 16. 6.

PROP. XVIII. PROB.

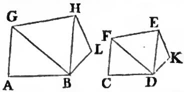
UPON a given ſtraight line to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated unto a given rectilineal figure.

Let AB be the given ſtraight line, and CDEF the given rectilineal figure of four ſides; it is required upon the given ſtraight line AB to deſcribe a rectilineal figure ſimilar and ſimilarly ſituated unto CDEF.



Join DF, and at the points A, B in the ſtraight line AB make [a] a. 23. 1. the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF; therefor the remaining angle CFD is equal to the remaining angle AGB[b]. b. 32. 1. wherefor the triangle FCD is equiangular to the triangle GAB. again, at the points G, B in the ſtraight line GB make [a] the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefor the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH. therefor becauſe the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH ſhall be equal to the whole CFE. by the ſame reaſon, the angle ABH is equal to the angle CDE; alſo the angle at A is equal to the angle at C, and the angle GHB to FED. therefor the rectilineal figure ABHG is equiangular to CDEF. but likewiſe theſe figures have their ſides about the

equal angles proportionals. becauſe the triangles GAB, FCD being
c. 4. 6. equiangular, BA ſhall [c] be to AG, as DC to CF; and becauſe AG is to GB, as CF to FD; and as GB to GH, ſo, by reaſon of the equi-
d. 22. 5. angular triangles BGH, DFE, is FD to FE; ex aequali [d], AG ſhall be to GH, as CF to FE. in the ſame manner it ſhall be proved that AB is to BH, as CD to DE. and GH is to HB, as FE to ED [c]. Wherefor becauſe the rectilineal figures ABHG, CDEF are equiangular, and have their ſides about the equal angles proportionals,
e. 1. Def. 6. they are ſimilar to one another [e].



Next, Let it be required to deſcribe upon a given ſtraight line AB, a rectilineal figure ſimilar, and ſimilarly ſituated unto the rectilineal figure CDKEF of five ſides.

Join DE, and upon the given ſtraight line AB deſcribe the rectilineal figure ABHG ſimilar and ſimilarly ſituated unto the quadrilateral figure CDEF, by the former caſe. and at the points B, H in the ſtraight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefor the remaining angle at K is equal to the remaining angle at L. and becauſe the figures ABHG, CDEF are ſimilar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefor the whole angle GHL is equal to the whole angle FEK. by the ſame reaſon, the angle ABL is equal to the angle CDK. therefor the five ſided figures AGHLB, CFEKD are equiangular. and becauſe the figures AGHB, CFED are ſimilar, GH is to HB, as FE to ED; and as HB to HL, ſo is ED to EK [c]; therefor ex aequali [d], GH is to HL, as FE to EK. by the ſame reaſon, AB is to BL, as CD to DK. and BL is to LH, as [c] DK to KE, becauſe the triangles

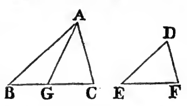
BLH, DKE are equiangular. therefor becauſe the five ſided figures AGHLB, CFEKD are equiangular, and have their ſides about the equal angles proportionals, they ſhall be ſimilar to one another. and in the ſame manner a ſix ſided rectilineal figure may be deſcribed upon a given ſtraight line ſimilar to one given, and ſo on. Which was to be done.

PROP. XIX. THEOR.

SIMILAR triangles are to one another in the duplicate ratio of their homologous ſides.

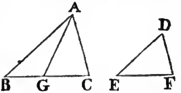
Let ABC, DEF be ſimilar triangles having the angle B equal to the angle E, and let AB be to BC, as DE to EF, ſo that the ſide BC be homologous to EF. the triangle ABC ſhall have to the triangle DEF, the duplicate ratio of that which BC has to EF.

Take BG a third proportional to BC, EF[a], ſo that BC be to EF, a. 11. 6. as EF to BG, and join GA. then, becauſe as AB to BC, ſo DE to EF; alternately[b], AB ſhall be to b. 16. 5. DE, as BC to EF. but as BC to EF, ſo is EF to BG; therefor[c] c. 11. 5. as AB to DE, ſo is EF to BG. wherefor the ſides of the triangles ABG, DEF which are about the equal angles are reciprocally proportional. but triangles which have the ſides about two equal angles reciprocally proportional are equal to one another[d]. therefor the triangle ABG is equal to the triangle DEF. d. 15. 6. and becauſe as BC is to EF, ſo EF to BG; and if three ſtraight lines be proportionals, the firſt is ſaid[e] to have to the third the duplicate e. 10. Def. 5. ratio of that which it has to the ſecond; BC therefor ſhall have to



BG the duplicate ratio of that which BC has to EF. but as BC to BG, ſo is [f] the triangle ABC to the triangle ABG. therefor the triangle ABC has to the triangle ABG, the duplicate ratio of that which BC has to EF. but the triangle ABG is equal to the triangle DEF; wherefor alſo the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. therefor ſimilar triangles &c. Q. E. D.

f. 1. 6.



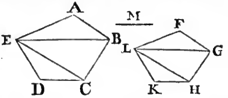
Cor. From this it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any triangle upon the firſt to a ſimilar and ſimilarly deſcribed triangle upon the ſecond.

PROP. XX. THEOR.

SIMILAR Polygons are divided into the ſame number of ſimilar triangles, which have the ſame ratio that the whole Polygons have to one another; and the polygons have to one another the duplicate ratio of that which their homologous ſides have.

Let ABCDE, FGHKL be ſimilar polygons, and let AB be the homologous ſide to FG. the polygons ABCDE, FGHKL can be divided into the ſame number of ſimilar triangles, whereof each to each ſhall have the ſame ratio which the polygons have; and the polygon ABCDE ſhall have to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the ſide FG.

Join BE, EC, GL, LH. and becauſe the polygon ABCDE is ſimilar to the polygon FGHKL, the angle BAE is equal to the angle GFL [a], GFL[a], and BA is to AE, as GF to FL[a]. wherefor becauſe the triangles ABE, FGL have an angle in one equal to an angle in the other, and their ſides about theſe equal angles proportionals, the triangle ABE ſhall be equiangular[b], and therefor ſimilar to the triangle FGL[c]; wherefor the angle ABE is equal to the angle FGL. and, becauſe the polygons are ſimilar, the



whole angle ABC is equal [a] to the whole angle FGH; therefor the remaining angle EBC is equal to the remaining angle LGH. and becauſe the triangles ABE, FGL are ſimilar, EB is to BA, as LG to GF[a]; and alſo, becauſe the polygons are ſimilar, AB is to BC, as FG to GH[a]; therefor, ex aequali[d], EB is to BC, as LG to GH; that is, the ſides about the equal angles EBC, LGH are proportionals; therefor [b] the triangle EBC is equiangular to the triangle LGH, and ſimilar to it[c]. by the ſame reaſon the triangle ECD likewiſe is ſimilar to the triangle LHK. therefor the ſimilar polygons ABCDE, FGHKL are divided into the ſame number of ſimilar triangles.

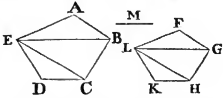
a. 1. Def. 6.
b. 6. 6.
c. 4. 6.
d. 22. 5.

Alſo theſe triangles ſhall have, each to each, the ſame ratio which the whole polygons have to one another, the antecedents being ABE, EBC, ECD, and the conſequents FGL, LGH, LHK. and the polygon ABCDE ſhall have to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the homologous ſide FG.

Becauſe the triangle ABE is ſimilar to the triangle FGL, ABE ſhall have to FGL the duplicate ratio [e] of that which the ſide BE has to the ſide GL. by the ſame reaſon, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL. therefor as the triangle ABE to the triangle FGL, ſo [f] is the triangle BEC to the triangle GLH.

e. 19. 6.
f. 11. 5.

Again, becauſe the triangle EBC is ſimilar to the triangle LGH, EBC ſhall have to LGH, the duplicate ratio of that which the ſide EC has to the ſide LH. by the ſame reaſon, the triangle ECD ſhall have to the triangle LHK, the duplicate ratio of that which EC has to LH. as therefor the triangle EBC to the triangle LGH, ſo
f. 11. 5. is [f] the triangle ECD to the triangle LHK. but it has been proved that likewiſe the triangle EBC is to the triangle LGH, as the triangle ABE to the triangle FGL. Therefor as the triangle ABE is to the triangle FGL, ſo triangle EBC to triangle LGH, and triangle ECD to triangle LHK. and therefor as one of the antecedents to one of the conſe-
g. 12. 5. quents, ſo are all the antecedents to all the conſequents[g]. Wherefor as the triangle ABE to the triangle FGL, ſo is the polygon ABCDE to the polygon FGHKL. but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the ſide AB has to the homologous ſide FG. Therefor alſo the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous ſide FG. Wherefor ſimilar polygons &c. Q. E. D.



Cor. 1. In like manner it may be proved that ſimilar four ſided figures, or of any number of ſides are one to another in the duplicate ratio of their homologous ſides, and it has already been proved in triangles. Therefor univerſally, ſimilar rectilineal figures are to one another in the duplicate ratio of their homologous ſides.

Cor. 2. And if to AB, FG two of the homologous ſides a third
h. 10. Def. 5. proportional M be taken, AB ſhall have [h] to M the duplicate ratio of that which AB has to FG. but the four ſided figure or polygon upon

AB has to the four ſided figure or polygon upon FG likewiſe the duplicate ratio of that which AB has to FG. therefor as AB is to M, ſo is the figure upon AB to the figure upon FG, which alſo was proved in triangles[i]. Therefor, univerſally, it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any rectilineal figure upon the firſt, to a ſimilar and ſimilarly deſcribed rectilineal figure upon the ſecond.

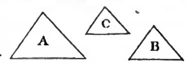
i. Cor. 19. 6.

PROP. XXI. THEOR.

RECTILINEAL figures which are ſimilar to the ſame rectilineal figure, are alſo ſimilar to one another.

Let each of the rectilineal figures A, B be ſimilar to the rectilineal figure C. the figure A ſhall be ſimilar to the figure B.

Becauſe A is ſimilar to C, they ſhall be equiangular, and alſo have their ſides about the equal angles proportionals[a]. again, becauſe B is ſimilar to C, they ſhall be equiangular, and have their ſides about the equal angles proportionals[a]. therefor the figures A, B are each of them equiangular to C, and have the ſides about the equal angles of each of them and of C proportionals. Wherefor the rectilineal figures A and B are equiangular[b], and have their ſides about the equal angles proportionals[c]. Therefor A is ſimilar[a] to B. Q. E. D.



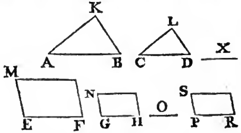
a. 1. Def. 6.
b. 1. Ax. 1.
c. 11. 5.

PROP. XXII. THEOR.

IF four ſtraight lines be proportionals, the ſimilar rectilineal figures ſimilarly deſcribed upon them ſhall alſo be proportionals. and if the ſimilar rectilineal figures ſimilarly deſcribed upon four ſtraight lines be proportionals, theſe ſtraight lines ſhall be proportionals.

Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH, and upon AB, CD let the ſimilar rectilineal figures KAB, LCD be ſimilarly deſcribed; and upon EF, GH the ſimilar rectilineal figures MF, NH, in like manner. the rectilineal figure KAB ſhall be to LCD, as MF to NH.

a. 11. 6. To AB, CD take a third proportional [a] X; and to EF, GH a third proportional O. and becauſe AB is to CD, as EF to GH, alſo



b. 11. 5. CD ſhall be [b] to X, as GH to O; wherefor ex aequali [c], as AB to X,
c. 22. 5.
d. 1. Cor. 20. 6. ſo EF to O. but as AB to X, ſo is [d] the rectilineal KAB to the rectilineal LCD, and as EF to O, ſo is [d] the rectilineal MF to the rectilineal NH. therefor as KAB to LCD, ſo [e] is MF to NH.

And if the rectilineal KAB be to LCD, as MF to NH; the ſtraight line AB ſhall be ſo CD, as EF to GH.

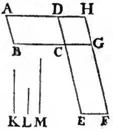
Make [e] as AB to CD, ſo EF to PR, and upon PR deſcribe [f] the rectilineal figure SR ſimilar and ſimilarly ſituated to either of the figures MF, NH. therefor becauſe as AB to CD, ſo EF to PR, and that upon AB, CD are deſcribed the ſimilar and ſimilarly ſituated rectilineals KAB, LCD, and upon EF, PR, in like manner, the ſimilar rectilineals MF, SR; KAB ſhall be to LCD, as MF to SR; but, by the Hypotheſis, KAB is to LCD, as MF to NH; and therefor the rectilineal MF having the ſame ratio to each of the two NH, SR, they ſhall be equal [g] to one another. they are alſo ſimilar, and ſimilarly ſituated; therefor GH is equal to PR. and becauſe as AB to CD, ſo is EF to PR, and that PR is equal to GH; AB ſhall be to CD, as EF to GH. If therefor four ſtraight lines &c. Q. E. D.

e. 12. 6.
f. 18. 6.
g. 9. 5.

PROP. XXIII. THEOR.

EQUIANGULAR parallelograms have to one another the ratio which is compounded of the ratios of their ſides.

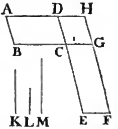
Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG. the ratio of the parallelogram AC to the parallelogram CF, ſhall be the ſame with the ratio which is compounded of the ratios of their ſides.



Let BC, CG be placed in a ſtraight line, then ſhall DC and CE be alſo in a ſtraight line[a]; and compleat the parallelogram DG,

a. 14. 1.

and, taking any ſtraight line K, make [b] as BC to CG, ſo K to L; and
b. 12. 6. as DC to CE, ſo make [b] L to M. therefor the ratios of K to L, and L to M are the ſame with the ratios of the ſides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is ſaid to be
c. A. Def. 5. compounded [c] of the ratios of K to L, and L to M. wherefor alſo K has to M, the ratio compounded of the ratios of the ſides. and becauſe as BC to CG, ſo is the parallelogram AC to the parallelogram
d. 1. 6. CH[d]; but as BC to CG, ſo is K to L; K
e. 11. 5. ſhall be [e] to L, as the parallelogram AC to the parallelogram CH. again, becauſe as DC to CE, ſo is the parallelogram CH to the parallelogram CF; but as DC to CE, ſo is L to M; L ſhall [e] be to M, as the parallelogram CH to the parallelogram CF. therefor ſince it has been proved that as K to L, ſo is the parallelogram AC to the parallelogram CH; and as L to M, ſo the
f. 22. 5. parallelogram CH to the parallelogram CF; ex aequali[f], K ſhall be to M, as the parallelogram AC to the parallelogram CF. but K has to M the ratio which is compounded of the ratios of the ſides; therefor alſo the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the ſides. Wherefor equiangular parallelograms &c. Q. E. D.



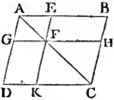
PROP. XXIV. THEOR.

THE parallelograms about the diameter of any parallelogram, are ſimilar to the whole, and to one another.

Let ABCD be a parallelogram, whoſe diameter is AC; and EG,
HK

HK the parallelograms about the diameter. the parallelograms EG, HK ſhall be ſimilar both to the whole parallelogram ABCD, and to one another.

Becauſe DC, GF are parallels, the angle ADC ſhall be equal [a] to a. 29. 1.
the angle AGF. by the ſame reaſon, becauſe BC, EF are parallels,
the angle ABC is equal to the angle AEF. and each of the angles
BCD, EFG is equal to the oppoſite angle DAB[b], b. 34. 1.
and therefor are equal to one another; wherefor the parallelograms ABCD, AEFG are equiangular. and becauſe the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they ſhall be equiangular to one another; therefor [c] as AB to BC, ſo is AE to EF. c. 4. 6.
and becauſe the oppoſite ſides of parallelograms are equal to one another[b], AB ſhall [d] be to AD, as AE to AG; and DC to CB, as GF d. 7. 5.
to FE; and alſo CD to DA, as FG to GA. therefor the ſides of
the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefor ſimilar to one another[e]. by the ſame e. 1. Def. 6.
reaſon, the parallelogram ABCD is ſimilar to the parallelogram FHCK.
Wherefor each of the parallelograms GE, KH is ſimilar to DB. but
rectilineal figures which are ſimilar to the ſame rectilineal figure, are
alſo ſimilar to one another[f], therefor the parallelogram GE is ſimilar f. 21. 6.
to KH. Wherefor in every &c. Q. E. D.

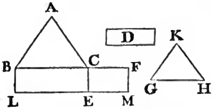


PROP. XXV. PROB.

TO deſcribe a rectilineal figure which ſhall be ſimilar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be deſcribed is required to be ſimilar, and D that to which it muſt be equal. It is required to deſcribe a rectilineal figure ſimilar to ABC and equal to D.

a. Cor. 45. 1. Upon the ſtraight line BC deſcribe [a] the parallelogram BE equal to the figure ABC; and upon CE deſcribe [a] the parallelogram CM equal to D, having the angle FCE equal to the angle CBL. therefor BC b. 29. 1. 14. 1. and CF are in a ſtraight line [b], as alſo LE and EM. between BC and c. 13. 6. CF find [c] a mean proportional GH, and upon GH deſcribe [d] the d. 18. 6. rectilineal figure KGH ſimilar and ſimilarly ſituated to the figure ABC. and becauſe BC is to GH, as GH to CF, and if three ſtraight lines be e. 2. Cor. 20. 6. proportionals, as the firſt is to the third, ſo is [e] the figure upon the firſt to the ſimilar and ſimilarly deſcribed figure upon the ſecond; therefor as BC to CF, ſo ſhall the rectilineal figure ABC be to KGH. but as f. 1. 6. BC to CF, ſo is [f] the parallelogram BE to the parallelogram EF. therefor as the rectilineal figure ABC is to KGH, ſo is the parallelo- g. 11. 5. gram BE to the parallelogram EF [g]. and the rectilineal figure ABC is equal to the parallelogram BE; therefor the rectilineal figure KGH is h. 14. 5. equal [h] to the parallelogram EF. but EF is equal to the figure D, wherefor alſo KGH is equal to D; and it is ſimilar to ABC. Therefor the rectilineal figure KGH has been deſcribed ſimilar to the figure ABC, and equal to D. Which was to be done.

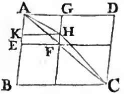


PROP. XXVI.

PROP. XXVI. THEOR.

IF two ſimilar parallelograms have a common angle, and be ſimilarly ſituated, they ſhall be about the ſame diameter.

Let the parallelograms ABCD, AEFG be ſimilar and ſimilarly ſituated, and have the angle DAB common. ABCD and AEFG ſhall be about the ſame diameter.



For if not, let, if poſſible, the parallelogram BD have its diameter AHC in a different ſtraight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and thro' H draw HK parallel to AD or BC. therefor becauſe the parallelograms ABCD, AKHG are about the ſame diameter, they ſhall be ſimilar to one another[a]. wherefor as DA to AB, ſo is [b] GA to AK. but becauſe ABCD and AEFG are ſimilar parallelograms, as DA is to AB, ſo GA to AE. therefor [c] as GA to AE, ſo GA to AK; wherefor GA has the ſame ratio to each of the ſtraight lines AE, AK; and AK ſhall be equal [d] to AE, the leſſer to the greater, which is impoſſible. therefor ABCD and AKHG are not about the ſame diameter; wherefor ABCD and AEFG muſt be about the ſame diameter. Therefor if two ſimilar &c. Q. E. D.

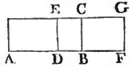
a. 24. 6.
b. 1. Def. 6.
c. 11. 5.
d. 9. 5.

' To underſtand the three following propoſitions more eaſily, it is ' to be obſerved,

' 1. That a parallelogram is ſaid to be applied to a ſtraight line, ' when it is deſcribed upon it as one of its ſides. Ex. gr. the parallelogram AC is ſaid to be applied to the ſtraight line AB.

' 2.

'2. But a parallelogram AE is ſaid to be applied to a ſtraight line AB, deficient by a parallelogram, when AD the baſe of AE is leſs than AB, and therefor AE is leſs than the parallelogram AC deſcribed upon AB in the ſame angle, and between the ſame parallels, by the parallelogram DC; and DC is therefor called the defect of AE.



'3. And a parallelogram AG is ſaid to be applied to a ſtraight line AB, exceeding by a parallelogram, when AF the baſe of AG is greater than AB, and therefor AG exceeds AC the parallelogram deſcribed upon AB in the ſame angle, and between the ſame parallels, by the parallelogram BG.'

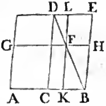
PROP. XXVII. THEOR.

OF all parallelograms applied to the ſame ſtraight line, and deficient by parallelograms ſimilar and ſimilarly ſituated to that which is deſcribed upon the half of the line; the greateſt is that which is applied to the half, which is ſimilar to its defect.

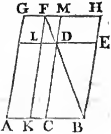
Let AB be a ſtraight line divided into two equal parts in C; and let the parallelogram AD be applied to the half AC, which is therefor deficient from the parallelogram upon the whole line AB by the parallelogram CE upon the other half CB. the parallelogram AD ſhall be the greateſt of all the parallelograms applied to any other parts of AB which are deficient by parallelograms that are ſimilar and ſimilarly ſituated to CE.

Let AF be any parallelogram applied to AK any other part of AB than the half, ſo as to be deficient from the parallelogram upon the whole line AB by the parallelogram KH ſimilar and ſimilarly ſituated to CE; AD ſhall be greater than AF.

Firſt, Let AK the baſe of AF be greater than AC the half of
AB; and becauſe CE is ſimilar to the parallelogram KH, they ſhall be about the ſame diameter[a]. draw their diameter DB, and compleat a. 26. 6.
the ſcheme. therefor becauſe the parallelogram
CF is equal [b] to FE, add KH to both, and the b. 43. 1.
whole CH ſhall be equal to the whole KE. but
CH is equal [c] to CG, becauſe the baſe AC is c. 36. 1.
equal to the baſe CB; therefor CG is equal to KE. to each of theſe add CF; the whole AF ſhall be equal to the gnomon CHL. therefor CE or the parallelogram AD is greater than the parallelogram AF.



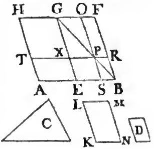
Next, Let AK the baſe of AF be leſſer than AC, and, the ſame conſtruction being made, the parallelogram DH ſhall be equal to DG[c], for
HM is equal to MG[d], becauſe BC is equal to d. 34. 1.
CA; wherefor DH ſhall be greater than LG. but DH is equal [b] to DK; therefor DK is greater than LG. to each of theſe add AL; and the whole AD ſhall be greater than the whole AF. Therefor the parallelogram applied &c. Q. E. D.



PROP. XXVIII. PROB.

TO a given ſtraight line to apply a parallelogram that ſhall be equal to a given rectilineal figure, and be deficient by a parallelogram ſimilar to a given parallelogram. but the given rectilineal figure to which the parallelogram to be applied is to be equal, muſt not be greater than the parallelogram applied to the half of the given line whoſe defect is ſimilar to the defect of that which is to be applied; that is, to the given parallelogram.

Let AB be the given ſtraight line, and C the given rectilineal figure, to which the parallelogram to be applied is required to be equal, which figure muſt not be greater than the parallelogram applied to the half of the line having its defect from that upon the whole line ſimilar to the defect of that which is to be applied; and let D be the parallelogram to which this defect is required to be ſimilar. It is required to apply a parallelogram to the ſtraight line AB, which ſhall be equal to the figure C, and be deficient from the parallelogram upon the whole line by a parallelogram ſimilar to D.



a. 10. 1. Divide AB into two equal parts [a] in
b. 18. 6. the point E, and upon EB deſcribe the parallelogram EBFG ſimilar [b] and ſimilarly ſituated to D, and compleat the parallelogram AG, which muſt either be equal to C, or greater than it, by the determination. and if AG be equal to C, then what was required is already done;

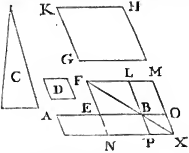
done; for upon the ſtraight line AB the parallelogram AG is applied
equal to the figure C, and deficient by the parallelogram EF ſimilar
to D. but if AG be not equal to C, it is greater than it; and
EF is equal to AG, therefor EF alſo is greater than C. Make [c] the c. 25. 6.
parallelogram KLMN equal to the exceſs of EF above C, and ſimilar
and ſimilarly ſituated to D; but D is ſimilar to EF, therefor [d] alſo d. 21. 6.
KM is ſimilar to EF. let KL be the homologous ſide to EG, and
LM to GF. and becauſe EF is equal to C and KM together, EF
ſhall be greater than KM; therefor the ſtraight line EG is greater
than KL, and GF than LM. make GX equal to LK, and GO equal
to LM, and compleat the parallelogram XGOP. therefor XO is equal
and ſimilar to KM; but KM is ſimilar to EF; wherefor alſo XO is
ſimilar to EF, and therefor XO and EF are about the ſame diameter[e].
let GPB be their diameter, and compleat the ſcheme. then becauſe e. 26. 6.
EF is equal to C and KM together, and XO a part of the one
is equal to KM a part of the other, the remainder, viz. the gnomon
ERO ſhall be equal to the remainder C. and becauſe OR is equal [f] f. 34. 1.
to XS, by adding SR to each, the whole OB ſhall be equal to the
whole XB. but XB is equal [g] to TE, becauſe the baſe AE is equal to g. 36. 1.
the baſe EB; wherefor alſo TE is equal to OB. add XS to each,
and the whole TS ſhall be equal to the whole, viz. to the gnomon
ERO. but it has been proved that the gnomon ERO is equal to C,
and therefor alſo TS ſhall be equal to C. Wherefor the parallelogram
TS equal to the given rectilineal figure C, is applied to the given ſtraight
line AB deficient by the parallelogram SR ſimilar to the given one D,
becauſe SR is ſimilar to EF[h]. Which was to be done. h. 24. 6.

PROP. XXIX.

PROP. XXIX. PROB.

TO a given ſtraight line to apply a parallelogram that ſhall be equal to a given rectilineal figure, exceeding by a parallelogram ſimilar to another given.

Let AB be the given ſtraight line, and C the given rectilineal figure to which the parallelogram to be applied is required to be equal, and D the parallelogram to which the exceſs of the one to be applied above that upon the given line is required to be ſimilar. It is required to apply a parallelogram to the given ſtraight line AB which ſhall be equal to the figure C, exceeding by a parallelogram ſimilar to D.



Divide AB into two equal parts in the point E, and upon EB
a. 18. 6. deſcribe [a] the parallelogram EL ſimilar and ſimilarly ſituated to
b. 25. 6. D. and make [b] the parallelogram GH equal to EL and C together, and ſimilar and ſimilarly ſituated to D; wherefor GH ſhall be ſimilar
c. 21. 6. to EL[c]. let KH be the ſide homologous to FL, and KG to FE. and becauſe the parallelogram GH is greater than EL, the ſide KH ſhall be greater than FL, and KG than FE. produce FL and FE, and make FLM equal to KH, and FEN to KG, and compleat the parallelogram MN. MN is therefor equal and ſimilar to GH; but GH is ſimilar to EL; wherefor MN is ſimilar to EL, and EL and MN ſhall
d. 26. 6. be about the ſame diameter[d]. draw their diameter FX, and compleat the ſcheme. becauſe therefor GH is equal to EL and C together, and that

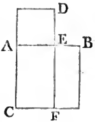
that GH is equal to MN; MN ſhall be equal to EL and C. take away the common part EL; the remainder, viz. the gnomon NOL ſhall be equal to C. and becauſe AE is equal to EB, the parallelogram AN ſhall be equal [e] to the parallelogram NB, that is to BM[f]. e. 36. 1. add NO to each; and the whole, viz. the parallelogram AX ſhall be f. 43. 1. equal to the gnomon NOL. but the gnomon NOL is equal to C; therefor alſo AX is equal to C. Wherefor to the ſtraight line AB there is applied the parallelogram AX equal to the given rectilineal C, exceeding by the parallelogram PO, which is ſimilar to D, becauſe PO is ſimilar to EL[g]. Which was to be done. g. 24. 6.

PROP. XXX. PROB.

TO cut a given ſtraight line in extreme and mean ratio.

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Upon AB deſcribe [a] the ſquare BC, and to AC apply the parallelogram CD equal to BC exceeding by the figure AD ſimilar to BC[b]. but BC is a ſquare, therefor alſo AD is a ſquare. and becauſe BC is equal to CD, by taking the common part CE from each, the remainder BF ſhall be equal to the remainder AD. and theſe figures are equiangular, therefor their ſides about the equal angles ſhall be reciprocally proportional[c]. wherefor as FE to ED, ſo AE to EB. but FE is equal to AC[d], that is to AB; and ED is equal to AE. therefor as BA to AE, ſo is AE to EB. but AB is greater than AE; wherefor AE is greater than EB[e]. a. 46. 1. b. 29. 6. c. 14. 6. d. 34. 1. e. 14. 5.

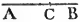


therefor the ſtraight line AB is cut in extreme and mean ratio in E[f].
f. 3. Def. 6. Which was to be done.

Otherwiſe,

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Divide AB in the point C, ſo that the rectangle contained by AB,
g. 11. 2. BC be equal to the ſquare of AC[g]. therefor becauſe the rectangle AB, BC is equal to the ſquare of AC,
h. 17. 6. as BA to AC, ſo ſhall AC be to CB[h]. therefor AB is cut in extreme and mean ratio in C[f]. Which was to be done.

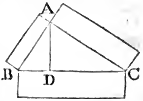


PROP. XXXI. THEOR.

IN right angled triangles the rectilineal figure deſcribed upon the ſide oppoſite to the right angle, is equal to the ſimilar, and ſimilarly deſcribed figures upon the ſides containing the right angle.

Let ABC be a right angled triangle, having the right angle BAC. the rectilineal figure deſcribed upon BC ſhall be equal to the ſimilar and ſimilarly deſcribed figures upon BA, AC.

Draw the perpendicular AD; therefor becauſe in the right angled triangle ABC, AD is drawn from the right angle at A perpendicular to the baſe BC, the triangles ABD, ADC ſhall be ſimilar to the whole triangle
a. 8. 6. ABC, and to one another[a]. and becauſe the triangle ABC is ſimilar to ABD, as
b. 4. 6. CB to BA, ſo ſhall BA be to BD[b]. and becauſe theſe three ſtraight lines are



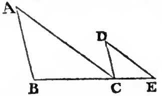
are proportionals, as the firſt to the third, ſo ſhall the figure upon the firſt be to the ſimilar, and ſimilarly deſcribed figure upon the ſecond[c]. c. 2. Cor. 20. 6. therefor as CB to BD, ſo is the figure upon CB to the ſimilar and ſimilarly deſcribed figure upon BA. and, inverſely[d], as DB to BC, ſo d. B. 5. is the figure upon BA to that upon BC. by the ſame reaſon, as DC to CB, ſo is the figure upon CA to that upon CB. Wherefor as BD and DC together to BC, ſo are the figures upon BA, AC to that upon BC[e]. but BD and DC together are equal to BC. Therefor the e. 24. 5. figure deſcribed on BC is equal[f] to the ſimilar and ſimilarly deſcribed f. A. 5. figures on BA, AC. wherefor in right angled triangles &c. Q. E. D.

PROP. XXXII. THEOR.

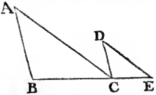
IF two triangles which have two ſides of the one proportional to two ſides of the other, be joined at one angle ſo as to have their homologous ſides parallel to one another; the remaining ſides ſhall be in a ſtraight line.

Let ABC, DCE be two triangles which have the two ſides BA, AC proportional to the two CD, DE, viz. BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE. BC and CE ſhall be in a ſtraight line.

Becauſe AB is parallel to DC, and the ſtraight line AC meets them, the alternate angles BAC, ACD ſhall be equal[a]; by the ſame reaſon the angle CDE is equal to the angle ACD; a. 29. 1. wherefor alſo BAC is equal to CDE. and becauſe the triangles ABC, DCE have one angle at A equal to one at D, and the ſides about theſe angles proportionals, viz. BA to AC, as CD



CD to DE, the triangle ABC ſhall be equiangular [b] to DCE. there-
b. 6. 6. for the angle ABC is equal to the angle DCE. and the angle BAC was proved to be equal to ACD. therefor the whole angle ACE is equal to the two angles ABC, BAC. add the common angle ACB, and the angles ACE, ACB ſhall be equal to the angles ABC, BAC, ACB. but ABC,
c. 32. 1. BAC, ACB are equal to two right angles[c]; therefor alſo the angles ACE, ACB are equal to two right angles. and ſince at the point C in the ſtraight line AC, the two ſtraight lines BC, CE, which are on the oppoſite ſides of it, make the adjacent angles ACE, ACB equal to two
d. 14. 1. right angles; therefor [d] BC and CE are in a ſtraight line. Wherefor if two triangles &c. Q. E. D.



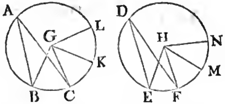
PROP. XXXIII. THEOR.

IN equal circles, angles whether at the centres or circumferences have the ſame ratio which the circumferences on which they ſtand have to one another. as alſo have the ſectors.

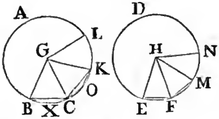
Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF at their circumferences. as the circumference BC to the circumference EF, ſo ſhall the angle BGC be to the angle EHF, and the angle BAC to the angle EDF; and alſo the ſector BGC to the ſector EHF.

Take any number of circumferences CK, KL each equal to BC, and any number whatever FM, MN each equal to EF; and join GK, GL, HM, HN. Becauſe the circumferences BC, CK, KL are all equal,

qual, the angles BGC, CGK, KGL ſhall alſo be all equal[a]. therefor what multiple ſoever the circumference BL is of the circumference a. 27. 3. BC, the ſame multiple is the angle BGL of the angle BGC. by the ſame reaſon, whatever multiple the circumference EN is of the circumference EF, the ſame multiple is the angle EHN of the angle EHF. and if the circumference BL be equal to the circumference EN, likewiſe the angle BGL ſhall be equal [a] to the angle EHN; and if the circumference BL be greater than EN, likewiſe the angle BGL ſhall be greater than EHN; and if leſs, leſſer. there being then four magnitudes, the two circumferences BC, EF, and the two angles BGC, EHF; of the circumference BC, and of the angle BGC, have been taken any multiples whatever, viz. the circumference BL, and the angle BGL; and of the circumference EF, and of the angle EHF, any equimultiples whatever, viz. the circumference EN, and the angle EHN. and it has been proved that if the circumference BL be greater than EN, the angle BGL is greater than EHN; and if equal, equal; and if leſs, leſſer. as therefor the circumference BC to the circumference EF, ſo [b] is the b. 5. Def. 5. angle BGC to the angle EHF. but as the angle BGC is to the angle EHF, ſo is [c] the angle BAC to the angle EDF, for each is double c. 15. 5. of each [d]. therefor as the circumference BC is to EF, ſo is the angle d. 20. 3. BGC to the angle EHF, and the angle BAC to the angle EDF.



Alſo, as the circumference BC to EF, ſo ſhall the ſector BGC be to the ſector EHF. Join BC, CK, and in the circumferences BC, CK take any points X, O, and join BX, XC, CO, OK. then becauſe in the triangles GBC, GCK the two ſides BG, GC are equal to the two CG, GK, and that they contain equal angles; the baſe BC ſhall be
e. 4. 1. equal [e] to the baſe CK, and the triangle GBC to the triangle GCK. and becauſe the circumference BC is equal to the circumference CK, the remaining part of the whole circumference of the circle ABC, ſhall be equal to the remaining part of the whole circumference of the
a. 27. 3. ſame circle. wherefor the angle BXC is equal to the angle COK [a];
f. 11. Def. 3. and the ſegment BXC is therefor ſimilar to the ſegment COK [f]; and



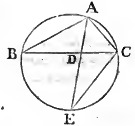
they are upon equal ſtraight lines BC, CK. but ſimilar ſegments of
g. 24. 3. circles upon equal ſtraight lines, are equal [g] to one another. therefor the ſegment BXC is equal to the ſegment COK. and the triangle BGC is equal to the triangle CGK; therefor the whole, the ſector BGC is equal to the whole, the ſector CGK. by the ſame reaſon the ſector KGL is equal to each of the ſectors BGC, CGK. in the ſame manner the ſectors EHF, FHM, MHN ſhall be proved equal to one another. therefor what multiple ſoever the circumference BL is of the circumference BC, the ſame multiple is the ſector BGL of the ſector BGC. by the ſame reaſon, whatever multiple the circumference EN is of EF, the ſame multiple is the ſector EHN of the ſector EHF. and

and if the circumference BL be equal to EN, the ſector BGL is equal to the ſector EHN; and if the circumference BL be greater than EN, the ſector BGL is greater than the ſector EHN; and if leſs, leſſer. ſince then there are four magnitudes, the two circumferences BC, EF, and the two ſectors BGC, EHF, and of the circumference BC and ſector BGC, the circumference BL and ſector BGL are any equimultiples whatever; and of the circumference EF and ſector EHF, the circumference EN and ſector EHN are any equimultiples whatever; and that it has been proved if the circumference BL be greater than EN, the ſector BGL is greater than the ſector EHN; and if equal, equal; and if leſs, leſſer. Therefor [b] as the circumference BC is to the circumference EF, ſo is the ſector BGC to the ſector EHF. Wherefor in equal circles &c. Q. E. D.

b. 5. Def. 5.

PROP. B. THEOR.

IF the angle of a triangle be biſſected by a ſtraight line, which likewiſe cuts the baſe; the rectangle contained by the ſides of the triangle ſhall be equal to the rectangle contained by the ſegments of the baſe, together with the ſquare of the ſtraight line which biſſects the angle.

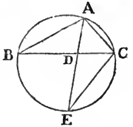


Let ABC be a triangle, and let the angle BAC be biſſected by the ſtraight line AD; the rectangle BA, AC ſhall be equal to the rectangle BD, DC together with the ſquare of AD.

Deſcribe the circle [a] ACB about the triangle, and produce AD to the circumference in E, and join EC. then becauſe the angle

a. 5. 4.

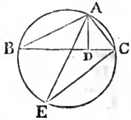
BAD is equal to the angle CAE, and the angle ABD to the angle [b] b. 21. 3. AEC, for they are in the ſame ſegment; the triangles ABD, AEC ſhall be equiangular to one another. therefor as BA to AD, c. 4. 6. ſo is [c] EA to AC, and the rectangle BA, d. 16. 6. AC ſhall be equal [d] to the rectangle EAD, e. 3. 2. that is [e] to the rectangle ED, DA together with the ſquare of AD. but the rectangle f. 35. 3. ED, DA is equal to the rectangle [f] BD, DC. Therefor the rectangle BA, AC is equal to the rectangle BD, DC together with the ſquare of AD. Wherefor if the angle &c. Q. E. D.



PROP. C. THEOR.

IF from an angle of a triangle a ſtraight line be drawn perpendicular to the baſe; the rectangle contained by the ſides of the triangle ſhall be equal to the rectangle contained by the perpendicular and the diameter of the circle deſcribed about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the baſe BC; the rectangle BA, AC ſhall be equal to the rectangle contained by AD and the diameter of the circle deſcribed about the triangle.



a. 5. 4. Deſcribe [a] the circle ACB about the triangle, and draw its diameter AE, and join EC. becauſe the right angle BDA is equal [b] to the angle ECA in a ſemicircle, and the angle ABD to the angle AEC in the ſame ſegment [c]; the triangles ABD, AEC are equiangular. therefor as [d] BA to AD, ſo is EA to AC, and the rectangle BA, AC ſhall be equal [e] to the rectangle EA, AD. If therefor from an angle &c. Q. E. D.

b. 31. 3.
c. 21. 3.
d. 4. 6.
e. 16. 6.

THE

THE
ELEMENTS
OF
EUCLID.

BOOK XI.

DEFINITIONS.

I.

A Solid is that which hath length, breadth, and thickneſs.

II.

That which bounds a ſolid is a ſuperficies.

III.

A ſtraight line is perpendicular, or at right angles, to a plane, when it makes right angles with every ſtraight line in that plane which meets it.

IV.

A plane is perpendicular to a plane, when the ſtraight lines in one of the planes which are perpendicular to the common ſection of the two planes, are perpendicular to the other plane.

The

V.

The inclination of a ſtraight line to a plane is the acute angle it contains with a ſtraight line drawn from the point in which the inclined line meets the plane, to the point in which a perpendicular to the plane, drawn from any point of the inclined line above the plane, meets the ſame plane.

VI.

The inclination of a plane to a plane is the acute angle contained by two ſtraight lines drawn from any the ſame point of their common ſection at right angles to it, one upon one plane, and the other upon the other plane.

VII.

Two planes are ſaid to have the ſame, or like inclination to one another, which two other planes have, when the ſaid angles of inclination are equal to one another.

VIII.

Parallel planes are ſuch which do not meet one another tho' produced.

IX.

A ſolid angle is that which is made by the meeting of more than two plane angles, which are not in the ſame plane, in one point.

X.

'The tenth Definition is omitted for reaſons given in the Notes.'

XI.

Similar ſolid figures are ſuch as have all their ſolid angles equal, each to each, and which are contained by the ſame number of ſimilar planes.

XII.

A Pyramid is a ſolid figure which is contained by planes that are conſtituted betwixt one plane and one point above it in which they meet.

XIII.

A Priſm is a ſolid figure contained by plane figures of which two that are oppoſite, are equal, ſimilar, and parallel to one another; and the others parallelograms.

XIV.

A Sphere is a ſolid figure deſcribed by the revolution of a ſemicircle about the diameter, which remains unmoved.

XV.

The axis of a ſphere is the fixed ſtraight line about which the ſemicircle revolves.

XVI.

The centre of a ſphere is the ſame with that of the ſemicircle.

XVII.

The diameter of a ſphere is any ſtraight line which paſſes through the centre, and is terminated both ways by the ſuperficies of the ſphere.

XVIII.

A Cone is a ſolid figure deſcribed by the revolution of a right angled triangle about one of the ſides containing the right angle, which ſide remains fixed.

If the fixed ſide be equal to the other ſide containing the right angle, the Cone is called a right angled Cone; if it be leſs than the other ſide, an obtuſe angled, and if greater, an acute angled Cone.

XIX.

The axis of a Cone is the fixed ſtraight line about which the triangle revolves.

XX.

The baſe of a Cone is the circle deſcribed by that ſide containing the right angle, which revolves.

XXI.

A Cylinder is a ſolid figure deſcribed by the revolution of a right angled parallelogram about one of its ſides which remains fixed.

XXII.

The axis of a cylinder is the fixed ſtraight line about which the parallellogram revolves.

XXIII.

The baſes of a cylinder are the circles deſcribed by the two revolving oppoſite ſides of the parallelogram.

XXIV.

Similar cones and cylinders are ſuch whoſe axes and the diameters of their baſes are proportionals.

XXV.

A Cube is a ſolid figure contained by ſix equal ſquares.

XXVI.

A Tetrahedron is a ſolid figure contained by four equal and equilateral triangles.

XXVII.

An Octahedron is a ſolid figure contained by eight equal and equilateral triangles.

XXVIII.

A Dodecahedron is a ſolid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

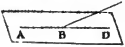
An Icoſahedron is a ſolid figure contained by twenty equal and equilateral triangles.

DEF. A.

A Parallelepiped is a ſolid figure contained by ſix quadrilateral figures whereof every oppoſite two are parallel.

PROP. I. THEOR.

ONE part of a ſtraight line cannot be in a plane and another part above it.

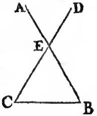


If it be poſſible, let AB part of the ſtraight line ABC be in the plane, and the part BC above it. and ſince the ſtraight line AB is in the plane it can be produced in that plane. let it be produced to D. therefor there are two ſtraight lines ABC, ABD that have a common ſegment AB, which is impoſſible[a].

a. Cor. 11. 1.

PROP. II. THEOR.

TWO ſtraight lines which cut one another are in one plane, and three ſtraight lines which meet one another are in one plane.



Let two ſtraight lines AB, CD cut one another in E; AB, CD ſhall be in one plane. and three ſtraight lines EC, CB, BE which meet one another, are in one plane.

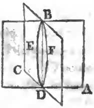
Let any plane paſs through the ſtraight line EB, and let the plane be turned about EB, produced if neceſſary, until it paſs thro' the point C. then becauſe the points E, C are in this plane, the ſtraight line EC ſhall be in it[a]. by the ſame reaſon, the ſtraight line BC ſhall be in the ſame; and, by the Hypotheſis EB is in it. therefor the three ſtraight lines EC, CB, BE are in one plane. but

a. 7. Def. 1.

in the plane in which EC, EB are, in the ſame are [b] CD, AB. therefor AB, CD are in one plane. Wherefor two ſtraight lines &c. Q. E. D. b. 1. 11.

PROP. III. THEOR.

IF two planes cut one another, their common ſection ſhall be a ſtraight line.



Let two planes AB, BC cut one another, and let the line DB be their common ſection; DB ſhall be a ſtraight line. If it be not, from the point D to B draw in the plane AB the ſtraight line DEB, and in the plane BC the ſtraight line DFB. then two ſtraight lines DEB, DFB have the ſame extremities, and therefor ſhall include ſpace betwixt them; which is impoſſible[a]. therefor BD the common ſection of the planes AB, BC cannot but be a ſtraight line. Wherefor if two planes &c. Q. E. D. a. 10. Ax. 1.

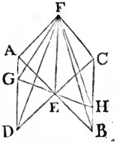
PROP. IV. THEOR.

IF a ſtraight line ſtand at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane which paſſes through them, i. e. the plane in which they are.

Let the ſtraight line EF ſtand at right angles to each of the ſtraight lines AB, CD in E the point of their interſection. EF ſhall alſo be at right angles to the plane paſſing thro' AB, CD.

Take the ſtraight lines AE, EB, CE, ED all equal to one another;

and thro' E draw, in the plane in which are AB, CD, any ſtraight line GEH; then from any point F in EF, draw FA, FG, FD, FC, FH, FB. and becauſe the two ſtraight lines AE, ED are equal to the two BE, EC, and that they contain equal angles [a] AED, BEC, the baſe AD ſhall be equal [b] to the baſe BC, and the angle DAE to the angle EBC. and the angle AEG is equal to the angle BEH [a]; therefor the triangles AEG, BEH have two angles of one equal to two angles of the other, each to each, and the ſides AE, EB, adjacent to the equal angles, equal to one another; wherefor they ſhall have their other ſides equal [c]. GE is therefor equal to EH, and AG to BH. and becauſe AE is equal to EB, and FE common and at right angles to them, the baſe AF ſhall be equal [b] to the baſe FB; by the ſame reaſon CF is equal to FD. and becauſe AD is equal to BC, and AF to FB, the two ſides FA, AD are equal to the two FB, BC, each to each; and the baſe DF was proved equal to the baſe FC; therefor the angle FAD is equal [d] to the angle FBC. again, it was proved that AG is equal to BH, and alſo AF to FB; FA then and AG, are equal to FB and BH, and the angle FAG has been proved equal to the angle FBH; therefor the baſe GF is equal [b] to the baſe FH. again, becauſe it was proved that GE is equal to EH, and EF is common; GE, EF ſhall be equal to HE, EF; and the baſe GF is equal to the baſe FH; therefor the angle GEF is equal [d] to the angle HEF, and each of theſe angles ſhall be a right [e] angle. Therefor FE makes right angles with GH any ſtraight line drawn thro' E in the plane paſſing thro' AB, CD. In like manner it ſhall be proved that FE makes right angles with every ſtraight line which meets it in that plane. But a ſtraight line is



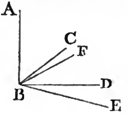
[a] 15. 1.
[b] 4. 1.
[c] 26. 1.
[d] 8. 1.
[e] 10. Def. 1.

at right angles to a plane when it makes right angles with every ſtraight line which meets it in that plane[f]. f. 3. Def. 11. therefor EF is at right angles to the plane in which are AB, CD. Wherefor if a ſtraight line &c. Q. E. D.

PROP. V. THEOR.

IF three ſtraight lines meet all in one point, and a ſtraight line ſtands at right angles to each of them in that point; theſe three ſtraight lines ſhall be in one and the ſame plane.

Let the ſtraight line AB ſtand at right angles to each of the ſtraight lines BC, BD, BE, in B the point where they meet; BC, BD, BE ſhall be in one and the ſame plane.

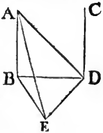


If not, let, if it be poſſible, BD and BE be in one plane, and BC be above it; and let a plane paſs through AB, BC, whoſe common ſection with the plane, in which BD and BE are, ſhall be a ſtraight [a] line; a. 3. 11. let this be BF. therefor the three ſtraight lines AB, BC, BF are all in one plane, viz. that which paſſes through AB, BC. and becauſe AB ſtands at right angles to each of the ſtraight lines BD, BE, it ſhall alſo be at right angles [b] to b. 4. 11. the plane paſſing through them; and therefor ſhall make right angles [c] c. 3. Def. 11. with every ſtraight line in that plane which meets it; but BF which is in that plane meets it. therefor the angle ABF is a right angle; but the angle ABC, by the Hypotheſis, is alſo a right angle; therefor the angle ABF is equal to the angle ABC, and they are both in the ſame plane, which is impoſſible. therefor the ſtraight line BC is not above the plane in which are BD and BE. wherefor the three ſtraight lines BC, BD, BE are in one and the ſame plane. therefor if three ſtraight lines &c. Q. E. D.

PROP. VI. THEOR.

IF two ſtraight lines be at right angles to the ſame plane, they ſhall be parallel to one another.

Let the ſtraight lines AB, CD be at right angles to the ſame plane; AB ſhall be parallel to CD.



Let them meet the plane in the points B, D, and draw the ſtraight line BD, to which draw DE at right angles, in the ſame plane; and make DE equal to AB, and join BE, AE, AD. then becauſe AB is perpendicular to the plane, it a. 3. 11. ſhall make right angles [a] with every ſtraight line which meets it, and is in that plane. but BD, BE, which are in that plane, do each of them meet AB. therefor each of the angles ABD, ABE is a right angle. by the ſame reaſon, each of the angles CDB, CDE is a right angle. and becauſe AB is equal to DE, and BD common, the two ſides AB, BD, are equal to the two ED, DB; and they contain right angles; therefor b. 4. 1. the baſe AD is equal [b] to the baſe BE. again, becauſe AB is equal to DE, and BE to AD; AB, BE are equal to ED, DA, and, in the triangles ABE, EDA, the baſe AE is common; therefor the angle c. 8. 1. ABE is equal [c] to the angle EDA. but ABE is a right angle; therefor EDA is alſo a right angle, and ED perpendicular to DA. but it is alſo perpendicular to each of the two BD, DC. wherefor ED is at right angles to each of the three ſtraight lines BD, DA, DC in the point in which they meet. therefor theſe three ſtraight lines are all in the ſame plane[d]. but AB is in the plane in which are BD, DA, because any three ſtraight lines which meet one another are in one plane[e]. therefor AB, BD, DC are in one plane. and each of the angles ABD, BDC is a right angle; therefor AB is parallel[f] to CD. Wherefor if two ſtraight lines &c. Q. E. D.

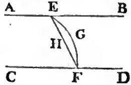
d. 5. 11.
e. 2. 11.
f. 28. 1.

PROP. VII. THEOR.

IF two ſtraight lines be parallel, the ſtraight line drawn from any point in the one to any point in the other ſhall be in the ſame plane in which the parallels are.

Let AB, CD be parallel ſtraight lines, and take any point E in the one, and the point F in the other. the ſtraight line which joins E and F ſhall be in the ſame plane in which the parallels are.

If not, let it be, if poſſible, above the plane, as EGF; and in the plane ABCD in which the parallels are, draw the ſtraight line EHF from E to F; and ſince EGF alſo is a ſtraight line, the two ſtraight lines EHF, EGF ſhall include ſpace betwixt them, which is impoſſible[a]. Therefor the ſtraight line joining the points E, F is not above the plane in which the parallels AB, CD are, and is therefor in that plane. Wherefor if two ſtraight lines &c. Q. E. D.



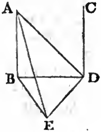
a. 10. Ax. 1.

PROP. VIII. THEOR.

If two ſtraight lines be parallel, and one of them is at right angles to a plane; the other alſo ſhall be at right angles to the ſame plane.

Let AB, CD be two parallel ſtraight lines, and let one of them AB be at right angles to a plane; the other CD ſhall be at right angles to the ſame plane.

Let AB, CD meet the plane in the points B, D, and join BD. therefor AB, CD, BD are in one plane. in the plane, to which AB is at right angles, draw DE at right angles to BD, and make DE equal to AB, and join BE, AE, AD. and becauſe AB is perpendicular to the plane, it ſhall be perpendicular to every ſtraight line which meets a. 3. Def. 11. it, and is in that plane[a]. therefor each of the angles ABD, ABE, is a right angle. and becauſe the ſtraight line BD meets the parallel ſtraight lines AB, CD, the angles ABD, CDB ſhall toge- b. 29. 1. ther be equal[b] to two right angles. and ABD is a right angle; therefor alſo CDB is a right angle, and CD perpendicular to BD. and becauſe AB is equal to DE, and BD common, the two AB, BD, are equal to the two ED, DB, and the angle ABD is equal to the angle EDB, becauſe each of them is a right angle; c. 4. 1. therefor the baſe AD is equal[c] to the baſe BE. again, becauſe AB is equal to DE, and BE to AD; the two AB, BE are equal to the two ED, DA; and the baſe AE is common to the triangles ABE, d. 8. 1. EDA; wherefor the angle ABE is equal[d] to the angle EDA. and ABE is a right angle; and therefor EDA is a right angle, and ED perpendicular

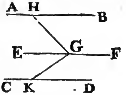


perpendicular to DA. but it is alſo perpendicular to BD; therefor ED is perpendicular [e] to the plane which paſſes through BD, DA, and e. 4. 11. ſhall [f] make right angles with every ſtraight line in that plane which f. 3. Def. 11. meets it. but DC is in the plane paſſing through BD, DA, becauſe all three are in the plane in which are the parallels AB, CD. wherefor ED is at right angles to DC; and therefor CD is at right angles to DE. but CD is alſo at right angles to DB; CD then is at right angles to the two ſtraight lines DE, DB in the point of their interſection D; and therefor is at right angles to the plane paſſing thro' DE, DB, which is the ſame plane to which AB is at right angles. Therefor if two ſtraight lines &c. Q. E. D.

PROP. IX. THEOR.

TWO ſtraight lines which are each of them parallel to the ſame ſtraight line, and not in the ſame plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the ſame plane with it; AB ſhall be parallel to CD.

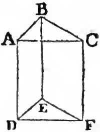


In EF take any point G, from which draw, in the plane paſſing through EF, AB, the ſtraight line GH at right angles to EF; and in the plane paſſing through EF, CD draw GK at right angles to the ſame EF. and becauſe EF is perpendicular both to GH and GK, EF ſhall be perpendicular [a] to the plane HGK paſſing a. 4. 11. through them. and EF is parallel to AB; therefor AB is at right angles [b] to the plane HGK. by the ſame reaſon, CD is likewiſe b. 8. 11. at right angles to the plane HGK. therefor AB, CD are each of them at right angles to the plane HGK. but if two ſtraight lines e. 6. 11. be at right angles to the ſame plane, they ſhall be parallel [e] to one another. therefor AB is parallel to CD. Wherefor if two ſtraight lines &c. Q. E. D.

PROP. X. THEOR.

IF two ſtraight lines which meet one another be parallel to two others that meet one another, and are not in the ſame plane with the firſt two; the firſt two and the other two ſhall contain equal angles.

Let the two ſtraight lines AB, BC which meet one another be parallel to the two ſtraight lines DE, EF that meet one another, and are not in the ſame plane with AB, BC. the angle ABC ſhall be equal to the angle DEF.



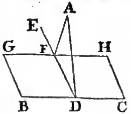
Take BA, BC, ED, EF all equal to one another; and join AD, CF, BE, AC, DF. becauſe therefor BA is equal and parallel to ED, AD a. 33. 1. ſhall [a] be both equal and parallel to BE. by the ſame reaſon, CF ſhall be equal and parallel to BE. therefor AD and CF are each of them equal and parallel to BE. but ſtraight lines that are parallel to the ſame ſtraight line, and not in the ſame plane with it, are parallel b. 9. 11. [b] to one another. therefor AD is parallel to CF; and it is equal c. 1. Ax. 1. [c] to it, and AC, DF join them towards the ſame parts; and therefor [a] AC is equal and parallel to DF. and becauſe AB, BC are equal to d. 8. 1. DE, EF, and the baſe AC to the baſe DF; the angle ABC ſhall be equal [d] to the angle DEF. Therefor if two ſtraight lines &c. Q. E. D.

PROP. XI.

PROP. XI. PROB.

TO draw a ſtraight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH. It is required to draw from the point A a ſtraight line perpendicular to the plane BH.



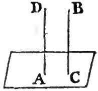
In the plane draw any ſtraight line BC, and from the point A draw [a] AD perpendicular to BC. If then AD be alſo perpendicular a. 12. 1. to the plane BH, the thing required is already done; but if it be not, from the point D draw [b] in the plane BH, b. 11. 1. the ſtraight line DE at right angles to BC; and from the point A draw AF perpendicular to DE; and through F draw [c] GH parallel to BC. c. 31. 1. and becauſe BC is at right angles to ED and DA, BC ſhall be at right angles [d] to the plane paſſing through ED, d. 4. 11. DA. and GH is parallel to BC; but if two ſtraight lines be parallel, one of which is at right angles to a plane, the other ſhall be at right angles [e] to the ſame plane; wherefor GH is at right angles to the e. 8. 11. plane through ED, DA, and ſhall be perpendicular [f] to every line in f. 3. Def. 11. that plane which meets it. but AF which is in the plane through ED, DA meets it. therefor GH is perpendicular to AF, and conſequently AF is perpendicular to GH. and AF is perpendicular to DE; therefor AF is perpendicular to each of the ſtraight lines GH, DE. but if a ſtraight line ſtands at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane paſſing through them. but the plane paſſing through ED, GH is the plane BH; therefor AF is perpendicular to the plane BH.

therefor from the given point A above the plane BH, the ſtraight line AF is drawn perpendicular to that plane. Which was to be done.

PROP. XII. PROB.

TO erect a ſtraight line at right angles to a given plane, from a point given in the plane.

Let A be the point given in the plane; it is required to erect a ſtraight line from the point A at right angles to the plane.

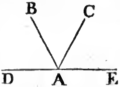


a. 11. 11. From any point B above the plane draw [a] b. 31. 1. BC perpendicular to it; and from A draw [b] AD parallel to BC. becauſe therefor AD, CB are two parallel ſtraight lines, and one of them BC is at right angles to the given plane, the other AD ſhall alſo be at right c. 8. 11. angles to it[c]. therefor a ſtraight line has been erected at right angles to a given plane from a point given in it. Which was to be done.

PROP. XIII. THEOR.

FROM the ſame point in a given plane there cannot be two ſtraight lines at right angles to the plane, upon the ſame ſide of it. and there can be but one perpendicular from a point above a plane, unto that plane.

For, if it be poſſible, let the two ſtraight lines AB, AC be at right angles to a given plane from the ſame point A in the plane, and upon the ſame ſide of it; and let a plane paſs through BA, AC; the a. 3. 11. common ſection of this with the given plane ſhall be a ſtraight [a] line paſſing through A. let DAE be their common ſection. therefor the ſtraight lines AB, AC, DAE are in one plane. and becauſe CA is at right angles to the given plane, it ſhall make right angles with every ſtraight line in that plane which meets it. but DAE which is in that plane meets CA; therefor CAE is a right angle. by the ſame reaſon BAE is a right angle. wherefor the angle CAE is equal to the angle BAE; and they are in one plane, which is impoſſible. and from a point above a plane there can be but one perpendicular unto that plane; for if there could be two, they would be parallel [b] to one another, which is abſurd. b. 6. 11. Therefor from the ſame point &c. Q. E. D.



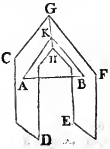
PROP. XIV. THEOR.

PLANES to which the ſame ſtraight line is perpendicular, are parallel to one another.

Let the ſtraight line AB be perpendicular to each of the planes CD, EF; theſe planes ſhall be parallel to one another.

If they be not, they ſhall meet one another when produced; let them meet; their common ſection ſhall be a ſtraight line GH, in which take any point K, and join AK, BK. then becauſe AB is perpendicular to the plane EF, it ſhall be perpendicular [a] to the ſtraight line BK which is in that plane. therefor ABK is a right angle. by the ſame reaſon, BAK is a right angle; wherefor the two angles ABK, BAK of the triangle ABK are equal to two right angles, which is impoſſible[b].

a. 3. Def. 11.



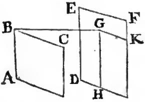
ſible[b]. therefor the planes CD, EF tho' produced do not meet one
b. 17. 1. another; that is, they are parallel[c]. Therefor planes &c. Q. E. D.
c. 8. Def. 11.

PROP. XV. THEOR.

If two ſtraight lines which meet one another, be parallel to two ſtraight lines which meet one another, but are not in the ſame plane with the firſt two; the planes which paſs through theſe ſtraight lines ſhall be parallel, i. e. the plane through the firſt two, and the plane through the other two.

Let AB, BC two ſtraight lines which meet one another, be parallel to DE, EF that meet one another, but are not in the ſame plane with AB, BC. the planes through AB, BC, and DE, EF ſhall not meet tho' produced.

* 11. 11. From the point B draw BG perpendicular * to the plane which paſ-
ſes through DE, EF, and let it meet that plane in G; and through
* 31. 1. G draw GH parallel * to ED, and GK parallel to EF. and becauſe
BG is perpendicular to the plane through
DE, EF, it ſhall make right angles with
every ſtraight line in that plane which
a. 3. Def. 11. meets it[a]. but the ſtraight lines GH,
GK in that plane meet it. therefor each
of the angles BGH, BGK is a right angle.
* 9. 11. and becauſe BA is parallel * to GH (for
each of them is parallel to DE, and are not both in the ſame plane
b. 29. 1. with it) the angles GBA, BGH are together equal [b] to two right
angles. and BGH is a right angle, therefor alſo GBA ſhall be a right
angle, and GB be perpendicular to BA. by the ſame reaſon, GB is
perpendicular



perpendicular to BC. therefor becauſe the ſtraight line GB ſtands at right angles to the two ſtraight lines BA, BC, that cut one another in B, GB ſhall be perpendicular [c] to the plane through BA, BC. and it is perpendicular to the plane through DE, EF; therefor BG is perpendicular to each of the planes through AB, BC and DE, EF. but thoſe planes to which the ſame ſtraight line is perpendicular, are parallel [d] to one another. therefor the plane through AB, BC is parallel to the plane through DE, EF. Wherefor if two ſtraight lines &c. Q. E. D.

c. 4. 11.

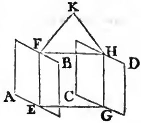
d. 14. 11.

PROP. XVI. THEOR.

IF two parallel planes be cut by another plane, their common ſections with it ſhall be parallel.

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common ſections with it be EF, GH. EF ſhall be parallel to GH.

For, if it is not, EF, GH ſhall meet, if produced, either on the ſide of FH, or EG. firſt let them be produced on the ſide of FH, and meet in the point K. therefor becauſe EFK is in the plane AB, every point in EFK ſhall be in that plane; and K is a point in EFK; therefor K is in the plane AB. by the ſame reaſon K is alſo in the plane CD. wherefor the planes AB, CD produced meet one another; but they do not meet, ſince they are parallel by the Hypotheſis. therefor the ſtraight lines EF, GH do not meet when produced on the ſide of FH. in the ſame manner it ſhall be proved that EF, GH do not



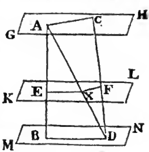
not meet when produced on the ſide of EG. but ſtraight lines which are in the ſame plane and do not meet, tho' produced either way, are parallel. therefor EF is parallel to GH. Wherefor if two parallel planes &c. Q. E. D.

PROP. XVII. THEOR.

IF two ſtraight lines be cut by parallel planes, they ſhall be cut in the ſame ratio.

Let the ſtraight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D. as AE is to EB, ſo ſhall CF be to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF. becauſe the two parallel planes KL, MN are cut by the plane EBDX, a. 16. 11. the common ſections EX, BD are parallel[a]. by the ſame reaſon, becauſe the two parallel planes GH, KL are cut by the plane AXFC, the common ſections AC, XF are parallel. and becauſe EX is parallel to BD a ſide of the triangle ABD, as AE to EB, b. 2. 6. ſo is [b] AX to XD. again, becauſe XF is parallel to AC a ſide of the triangle ADC, as AX to XD, ſo is CF c. 11. 5. to FD. and it was proved that AX is to XD, as AE to EB. therefor [c] as AE to EB, ſo is CF to FD. Wherefor if two ſtraight lines &c. Q. E. D.



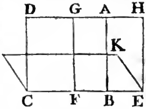
PROP. XVIII.

PROP. XVIII. THEOR.

IF a ſtraight line be at right angles to a plane, every plane which paſſes through it ſhall be at right angles to that plane.

Let the ſtraight line AB be at right angles to the plane CK. every plane which paſſes through AB ſhall be at right angles to the plane CK.

Let any plane DE paſs through AB, and let CE be the common ſection of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE at right angles to CE. and becauſe AB is perpendicular to the plane CK, it ſhall be perpendicular to every ſtraight line in that plane which meets it[a]. therefor it is perpendicular to a. 3. Def. 11. CE, and ABF ſhall be a right angle; but GFB is likewiſe a right angle; therefor AB is parallel [b] to FG. and AB is at right angles to the plane CK; b. 28. 1. therefor FG is alſo at right angles to the ſame plane[c]. but one plane c. 8. 11. is at right angles to another plane when the ſtraight lines in one of the planes, which are at right angles to their common ſection, are at right angles to the other plane[d]; and any ſtraight line FG in the d. 4. Def. 11. plane DE, which is at right angles to CE the common ſection of the planes, has been proved to be perpendicular to the other plane CK; therefor the plane DE is at right angles to the plane CK. In like manner, it ſhall be proved that all the planes which paſs through AB are at right angles to the plane CK. Therefor if a ſtraight line &c. Q. E. D.

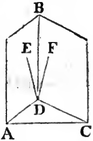


PROP. XIX. THEOR.

IF two planes which cut one another be each of them perpendicular to a third plane; their common ſection ſhall be perpendicular to the ſame plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common ſection of the firſt two. BD ſhall be perpendicular to the third plane.

If it be not, from the point D draw, in the plane AB, the ſtraight line DE at right angles to AD the common ſection of the plane AB with the third plane; and in the plane BC draw DF at right angles to CD the common ſection of the plane BC with the third plane. and becauſe the plane AB is perpendicular to the third plane, and DE is drawn in the plane AB at right angles to AD their common ſection, DE ſhall be perpendicular to the third plane[a]. in the ſame manner, it ſhall be proved that DF is perpendicular to the third plane. wherefor from the point D two ſtraight lines ſtand at right angles to the third plane, upon the ſame ſide of it, which is impoſſible[b]. therefor from the point D there cannot be any ſtraight line at right angles to the third plane, except BD the common ſection of the planes AB, BC. BD therefor is perpendicular to the third plane. Wherefor if two planes &c. Q. E. D.



[a] 4. Def. 11.

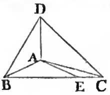
[b] 13. 11.

PROP. XX. THEOR.

IF a ſolid angle be contained by three plane angles, every two of them are greater than the third.

Let the ſolid angle at A be contained by the three plane angles BAC, CAD, DAB. every two of them ſhall be greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. but if they are not, let BAC be that angle which is not leſſer than any of the other two, and is greater than one of them DAB; and at the point A in the ſtraight line AB, make in the plane which paſſes thro'
BA, AC, the angle BAE equal [a] to the angle a. 23. 1.
DAB; and make AE equal to AD, and through E draw BEC cutting AB, AC in the points B, C, and join DB, DC. and becauſe DA is equal to AE, and AB is common, the two DA, AB are equal to the two EA, AB, and the angle DAB is equal
to the angle EAB. therefor the baſe DB is equal [b] to the baſe BE. b. 4. 1.
and becauſe BD, DC are greater [c] than CB, and one of them BD has c. 20. 1.
been proved equal to BE a part of CB, the other DC ſhall be greater than the remaining part EC. and becauſe DA is equal to AE, and AC common, but the baſe DC greater than the baſe EC; the angle DAC
ſhall be greater [d] than the angle EAC; and, by the conſtruction, the d. 25. 1.
angle DAB is equal to the angle BAE; wherefor the angles DAB, DAC are together greater than the angle BAC. but BAC is not leſſer than either of the angles DAB, DAC, therefor BAC with either of them is greater than the other. Wherefor if a ſolid angle &c. Q. E. D.

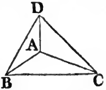


PROP. XXI. THEOR.

EVERY ſolid angle is contained by plane angles which together are leſs than four right angles.

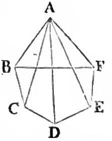
Firſt, Let the ſolid angle at A be contained by three plane angles BAC, CAD, DAB. theſe three ſhall together be leſſer than four right angles.

Take in each of the ſtraight lines AB, AC, AD any points B, C, D, and join BC, CD, DB. then, becauſe the ſolid angle at B is contained by the three plane angles CBA, ABD, DBC, any two of them a. 20. 11. whatever are greater [a] than the third; therefor the angles CBA, ABD are greater than the angle DBC. by the ſame reaſon, the angles BCA, ACD are greater than the angle DCB; and the angles CDA, ADB greater than BDC. wherefor the ſix angles CBA, ABD, BCA, ACD, CDA, ADB are greater than the three angles DBC, BCD, CDB. but the three angles DBC, BCD, CDB are equal to two right b. 32. 1. angles [b]. therefor the ſix angles CBA, ABD, BCA, ACD, CDA, ADB are greater than two right angles. and becauſe the three angles of each of the triangles ABC, ACD, ADB are equal to two right angles, the nine angles of theſe three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD, ſhall be equal to ſix right angles. of theſe the ſix angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles. therefor the remaining three angles BAC, DAC, BAD which contain the ſolid angle at A, are leſſer than four right angles.



But let the ſolid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; theſe ſhall together be leſs than four right angles.

Let the planes in which the angles are be cut by a plane, and let the common ſections of it with theſe planes be BC, CD, DE, EF, FB. and becauſe the ſolid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater [a] than the third, the angles CBA, ABF are greater than the angle FBC. by the ſame reaſon, the two plane angles at each of the points C, D, E, F, viz. thoſe angles which are at the baſe of the triangles, whoſe common vertex is A, ſhall be greater than the third angle at the ſame point, which is one of the angles of the polygon BCDEF. therefor all the angles at the baſes of the triangles are together greater than all the angles of the polygon. and becauſe all the angles of the triangles are together equal to twice as many right angles as there are triangles [b]; that is, as there are ſides in the polygon BCDEF; and that all the angles of the polygon together with four right angles are likewiſe equal to twice as many right angles as there are ſides in the polygon [c]; all the angles of the triangles ſhall be equal to all the angles of the polygon together with four right angles. But all the angles at the baſes of the triangles are greater than all the angles of the polygon, as has been proved, therefor the remaining angles of the triangles, viz. thoſe at the vertex, which contain the ſolid angle at A, are leſſer than four right angles. therefor every ſolid angle &c. Q. E. D.



a. 20. 11.

b. 32. 1.

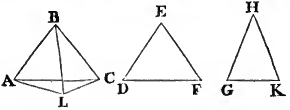
c. 1. Cor. 32. 1.

PROP. XXII. THEOR.

IF every two of three plane angles be greater than the third, and if the ſtraight lines which contain them be all equal; a triangle may be made of the ſtraight lines that join the extremities of theſe equal ſtraight lines.

Let ABC, DEF, GHK be three plane angles, whereof every two are greater than the third, and are contained by the equal ſtraight lines AB, BC, DE, EF, GH, HK; if their extremities be joined by the ſtraight lines AC, DF, GK, a triangle may be made of three ſtraight lines equal to AC, DF, GK; that is, every two of them ſhall be greater than the third.

a. 4. 1. If the angles at B, E, H are equal, AC, DF, GK ſhall be equal[a]; and any two of them greater than the third. but if the angles



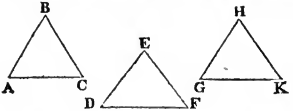
are not all equal, let the angle ABC be not leſſer than either of the two at E, H; therefor the ſtraight line AC ſhall not be leſ-
b. 4. or 24. 1. ſer than either of the other two DF, GK[b]; and it is plain that AC together with either of the other two ſhall be greater than the third. alſo DF with GK ſhall be greater than AC. for, at the
c. 23. 1. point B in the ſtraight line AB make[c] the angle ABL equal to the angle GHK, and make BL equal to one of the ſtraight lines AB, BC, DE, EF, GH, HK, and join AL, LC. then becauſe AB, BL are equal to GH, HK, and the angle ABL to the angle GHK, the baſe AL ſhall be equal to the baſe GK. and becauſe the angles at E, H are greater than the angle ABC, of which GHK is equal to ABL, the remaining angle at E ſhall be greater than the angle LBC. and becauſe the two ſides LB, BC are equal to the two DE, EF, and the angle DEF greater than the angle LBC, the baſe DF ſhall be
d. 24. 1. greater[d] than the baſe LC. and it has been proved that GK is equal to

to AL; therefor DF and GK are greater than AL and LC. but AL and LC are greater [e] than AC; much more then are DF and GK greater than AC. Wherefor every two of the ſtraight lines AC, DF, GK are greater than the third, and therefor a triangle can be made [f] whoſe ſides ſhall be equal to AC, DF, GK. Q. E. D. e. 20. 1. f. 22. 1.

PROP. XXIII. PROB.

TO make a ſolid angle contained by three given plane angles, every two of which is greater than the third, and all three together are leſſer than four right angles.

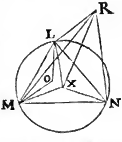
Let the three given plane angles be ABC, DEF, GHK, of which every two are greater than the third, and all of them together leſſer



than four right angles. It is required, to make a ſolid angle contained by three plane angles equal to ABC, DEF, GHK, each to each.

From the ſtraight lines which contain the angles, cut off AB, BC, DE, EF, GH, HK all equal to one another; and join AC, DF, GK. then a triangle can be made [a] of three ſtraight lines equal [b] to AC, DF, GK. let this be the triangle LMN, ſo that AC be equal to LM, DF to MN, and GK to LN; and about the triangle LMN deſcribe [c] a circle, and find its centre X, which will be either within the triangle, or in one of its ſides, or without it. a. 22. 11. b. 22. 1. c. 5. 4.

Firſt, Let the centre X be within the triangle, and join LX, MX, NX. AB ſhall be greater than LX. if it be not, AB muſt either be equal to LX or leſſer than it; firſt let it be equal. therefor becauſe AB is equal to LX, and AB is equal to BC, and LX to XM, AB and BC ſhall be equal to LX and XM, each to each; and the baſe AC is, by conſtruction, equal to the baſe LM; wherefor the angle ABC is equal to the angle LXM[d]. by the ſame reaſon the angle DEF is equal to the angle MXN, and the angle GHK to the angle NXL. therefor the three angles ABC, DEF, GHK are equal to the three angles LXM, MXN, NXL. but the three angles LXM, MXN, NXL are equal to four right angles[e]; therefor alſo the three angles ABC, DEF, GHK are equal to four right angles. but, by the Hypotheſis, they are leſſer than four right angles; which is abſurd. therefor AB is not equal to LX. but neither ſhall AB be leſſer than LX. for, if poſſible, let it be leſſer, and upon the ſtraight line LM, on the ſide of it on which is the centre X, deſcribe the triangle LOM, whoſe ſides LO, OM are equal to AB, BC; and becauſe the baſe LM is equal to the baſe AC, the angle LOM ſhall be equal to the angle ABC[d]. and AB, that is LO, by the Hypotheſis, is leſſer than LX; wherefor LO, OM ſhall fall within the triangle LXM; for, if they did fall upon its ſides, or without it, they ſhould be equal to, or greater than LX, XM[f]. therefor the angle LOM, that is, the angle ABC is greater than the angle LXM[f]. in the ſame manner it ſhall be proved, that the angle DEF is greater than the angle MXN, and the angle GHK greater than the angle NXL. therefor the three angles ABC, DEF, GHK are greater than the three angles LXM, MXN, NXL; that is, than four right angles.

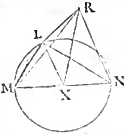


[d] 8. 1.

[e] 2. Cor. 15. 1.

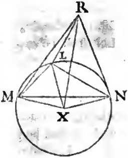
[f] 21. 1.

angles. but the ſame angles ABC, DEF, GHK are leſſer than four right angles; which is abſurd. therefor AB is not leſſer than LX, and it has been proved that it is not equal to LX; wherefor AB is greater than LX.



Next, Let the centre X of the circle fall in one of the ſides of the triangle, viz. in MN, and join XL. AB ſhall be greater than LX. for, if it be not, AB is either equal to LX or leſſer than it. firſt, let it be equal to LX. therefor AB and BC; that is, DE and EF are equal to MX and XL, that is to MN. but, by the conſtruction, MN is equal to DF; therefor DE, EF are equal to DF, which is impoſſible*. wherefor AB is not equal to LX; nor is it leſs; for then, much more, an abſurdity would follow. therefor AB is greater than LX.

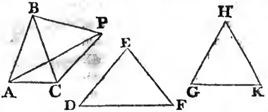
* 20. 1.

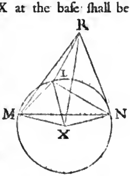


But let the centre X of the circle fall without the triangle LMN, and join LX, MX, NX. Likewiſe in this caſe AB ſhall be greater than LX. for if not, it is either equal to LX, or leſſer than it. firſt, let it be equal; it ſhall be proved, in the ſame manner as in the firſt caſe, that the angle ABC is equal to the angle MXL, and GHK to LXN; therefor the whole angle MXN is equal to the two angles ABC, GHK. but ABC and GHK are together greater than the angle DEF; therefor alſo the angle MXN is greater than DEF. and becauſe DE, EF are equal to MX, XN, and the baſe DF to the baſe MN, the angle MXN ſhall be equal [d] to the angle DEF. and it has been proved that it is greater than DEF, which is abſurd. therefor AB is not e-

d. 8. 1.

qual to LX. nor yet is it leſſer; for then, as has been proved in the firſt caſe, the angle ABC ſhall be greater than the angle MXL, and the angle GHK greater than the angle LXN. at the point B in the ſtraight line AB make the angle CBP equal to the angle GHK, and make BP equal to HK, and join CP, AP. and becauſe CB is equal to GH; CB, BP ſhall be equal to GH, HK, and they contain equal angles; wherefor the baſe CP is equal to the baſe GK, that is to LN. and in the Iſoſceles triangles ABC, MXL, becauſe the angle ABC is greater than the angle MXL, the angle MLX at the baſe ſhall be
g. 32. 1. greater [g] than the angle ACB at the baſe. by the ſame reaſon, becauſe the angle GHK, or CBP, is greater than the angle LXN, the angle XLN ſhall be greater than the angle BCP. therefor the whole angle MLN ſhall be greater than the whole angle ACP. and becauſe ML, LN are equal to AC, CP, each to each, but the angle MLN greater than the angle ACP, the
h. 24. 1. baſe MN ſhall be greater [h] than the baſe AP. and MN is equal to DF; therefor alſo DF is greater than AP. and becauſe DE, EF are equal to AB, BP, but the baſe DF greater than
k. 25. 1. the baſe AP, the angle DEF ſhall be greater [k] than the angle ABP. and ABP is equal to the two angles ABC, CBP, that is to the two angles





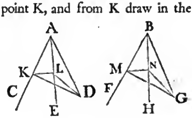
angles ABC, GHK; therefor the angle DEF is greater than the two angles ABC, GHK; but it is alſo leſſer than theſe two; which is impoſſible. therefor AB is not leſſer than LX; and it has been proved that it is not equal to it; therefor AB is greater than LX.

From the point X erect [l] XR at right angles to the plane of the l. 12. 11. circle LMN. and becauſe it has been proved in all the caſes, that AB is greater than LX, find a ſquare equal to the exceſs of the ſquare of AB above the ſquare of LX, and make RX equal to the ſide of it, and join RL, RM, RN. becauſe RX is perpendicular to the plane of the circle LMN, it ſhall [m] be perpendicular to each of the ſtraight m. 3. Def. 11. lines LX, MX, NX. and becauſe LX is equal to MX, and XR common, and at right angles to each of them, the baſe RL ſhall be equal to the baſe RM. by the ſame reaſon, RN is equal to each of the two RL, RM. therefor the three ſtraight lines RL, RM, RN are all equal. and becauſe the ſquare of XR is equal to the exceſs of the ſquare of AB above the ſquare of LX; the ſquare of AB ſhall be equal to the ſquares of LX, XR. but the ſquare of RL is equal [n] to the ſame n. 47. 1. ſquares, becauſe LXR is a right angle. therefor the ſquare of AB is equal to the ſquare of RL; and the ſtraight line AB to RL. but to AB each of BC, DE, EF, GH, HK is equal, and to RL each of the two RM, RN is equal. wherefor AB, BC, DE, EF, GH, HK are each of them equal to each of the ſtraight lines RL, RM, RN. and becauſe RL, RM, are equal to AB, BC, and the baſe LM to the baſe AC; the angle LRM ſhall be equal [o] to the angle ABC. by the o. 8. 1. ſame reaſon, the angle MRN is equal to the angle DEF, and NRL to GHK. Therefor there is made a ſolid angle at R, which is contained by three plane angles LRM, MRN, NRL, which are equal to the three given plane angles ABC, DEF, GHK, each to each. Which was to be done.

PROP. A. THEOR.

IF each of two ſolid angles be contained by three plane angles, which are equal to one another, each to each; the planes in which are the equal angles, have the ſame inclination to one another.

Let there be two ſolid angles at the points A, B; and let the angle at A be contained by the three plane angles CAD, CAE, EAD; and the angle at B by the three plane angles FBG, FBH, HBG; of which the angle CAD is equal to the angle FBG, and CAE to FBH, and EAD to HBG. the planes in which are the equal angles, ſhall have the ſame inclination to one another.



In the ſtraight line AC take any point K, and from K draw in the plane CAD the ſtraight line KD at right angles to AC, and in the plane CAE the ſtraight line KL at right angles to the ſame AC. therefor the angle DKL is the inclination [a] of the plane CAD to the plane CAE. in BF take BM equal to AK, and from the point M draw in the planes FBG, FBH the ſtraight lines MG, MN at right angles to BF; therefor the angle GMN is the inclination of the plane FBG to the plane FBH. join LD, NG; and becauſe in the triangles KAD, MBG, the angles KAD, MBG are equal, as alſo the right angles AKD, BMG, and the ſides AK, BM, adjacent to the equal angles, equal to one another, KD ſhall be equal [b] to MG, and AD to BG. by the ſame reaſon, in the triangles KAL, MBN, KL is equal to MN, and AL to BN. and in the triangles LAD, NBG, LA, AD

a. 6. Def. 11.

b. 26. 1.

AD are equal to NB, BG, and they contain equal angles; therefor the baſe LD is equal [c] to the baſe NG. laſtly, in the triangles KLD, MNG, the ſides DK, KL are equal to GM, MN, and the baſe LD to the baſe NG; therefor the angle DKL is equal [d] to the angle GMN. but the angle DKL is the inclination of the plane CAD to the plane CAE, and the angle GMN is the inclination of the plane FBG to the plane FBH, which planes have therefor the ſame inclination [e] to one another. and in the ſame manner it ſhall be demonſtrated, that the other planes in which the equal angles are, have the ſame inclination to one another. therefor if two ſolid angles &c. Q. E. D.

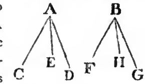
c. 4. 1.
d. 8. 1.
e. 7. Def. 11.

PROP. B. THEOR.

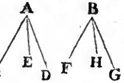
IF two ſolid angles be contained, each by three plane angles which are equal to one another, each to each, and alike ſituated; theſe ſolid angles ſhall be equal to one another.

Let there be two ſolid angles at A and B, of which the ſolid angle at A is contained by the three plane angles CAD, CAE, EAD; and that at B, by the three plane angles FBG, FBH, HBG; of which CAD is equal to FBG; CAE to FBH; and EAD to HBG. the ſolid angle at A, ſhall be equal to the ſolid angle at B.

Let the ſolid angle at A be applied to the ſolid angle at B; and firſt, the plane angle CAD being applied to the plane angle FBG, ſo as the point A may coincide with the point B, and the ſtraight line AC with BF, AD ſhall coincide with BG, becauſe the angle CAD is equal to the angle FBG. and becauſe the inclination of the plane CAE to



to the plane CAD is equal [a] to the inclination of the plane FBH to the plane FBG, the plane CAE ſhall coincide with the plane FBH, becauſe the planes CAD, FBG coincide with one another. and becauſe the ſtraight lines AC, BF coincide, and that the angle CAE is equal to the angle FBH, AE ſhall coincide with BH. and AD coincides with BG, wherefor the plane EAD coincides with the plane HBG. therefor the ſolid angle A coincides with the ſolid angle B, and conſequently they are equal [b] to one another. Q. E. D.

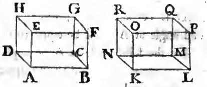


a. A. 11.
b. 8. Ax. 1.

PROP. C. THEOR.

SOLID figures which are contained by the ſame number of equal and ſimilar planes alike ſituated, and none of whoſe ſolid angles are contained by more than three plane angles; are equal and ſimilar to one another.

Let AG, KQ be two ſolid figures contained by the ſame number of ſimilar and equal planes, alike ſituated, viz. let the plane AC be ſimilar and equal to the plane KM; the plane AF to KP; BG to LQ; GD to QN; DE to NO; and laſtly, FH ſimilar and equal to PR. the ſolid figure AG ſhall be equal and ſimilar to the ſolid figure KQ.



Becauſe the ſolid angle at A is contained by the three plane angles BAD, BAE, EAD which, by the Hypotheſis, are equal to the plane angles LKN, LKO, OKN which contain the ſolid angle at K, each to each; the ſolid angle at A ſhall be equal [a] to the ſolid angle at K. in the ſame manner, the other ſolid angles of the figures are equal to one another. If then the ſolid figure AG be applied to the ſolid figure KQ, firſt the plane figure AC being applied to the plane figure KM; the ſtraight line AB coinciding with KL, the figure AC ſhall coincide with the figure KM, becauſe they are equal and ſimilar. therefor the ſtraight lines AD, DC, CB ſhall coincide with KN, NM, ML, each with each; and the points A, D, C, B with the points K, N, M, L. and the ſolid angle at A ſhall coincide [a] with the ſolid angle at K; wherefor the plane AF ſhall coincide with the plane KP, and the figure AF with the figure KP, becauſe they are equal and ſimilar unto one another. therefor the ſtraight lines AE, EF, FB ſhall coincide with KO, OP, PL; and the points E, F with the points O, P. In the ſame manner, the figure AH ſhall coincide with the figure KR, and the ſtraight line DH with NR, and the point H with the point R. and becauſe the ſolid angle at B is equal to the ſolid angle at L, it ſhall be proved in the ſame manner, that the figure BG ſhall coincide with the figure LQ, and the ſtraight line CG with MQ, and the point G with the point Q. therefor becauſe all the planes and ſides of the ſolid figure AG coincide with the planes and ſides of the ſolid figure KQ, AG ſhall be equal and ſimilar to KQ. and in the ſame manner, any other ſolid figures whatever which are contained by the ſame number of equal and ſimilar planes, alike ſituated, and none of whoſe ſolid angles are contained by more than three plane angles, ſhall be proved to be equal and ſimilar to one another. Q. E. D.

PROP. XXIV.

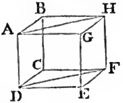
[a] B. 11.

PROP. XXIV. THEOR.

IF a ſolid be contained by ſix planes, two and two of which are parallel; the oppoſite planes ſhall be ſimilar and equal parallelograms.

Let the ſolid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE. its oppoſite planes ſhall be ſimilar and equal parallelograms.

Becauſe the two parallel planes BG, CE are cut by the plane AC, their
a. 16. 11. common ſections AB, CD are parallel[a]. again, becauſe the two parallel planes BF, AE are cut by the plane AC, their common ſections AD, BC are parallel[a]. and AB is parallel to CD; therefor AC is a parallelogram. in like manner, it ſhall be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram. join AH, DF; and becauſe AB is parallel to DC, and BH to CF; the two ſtraight lines AB, BH, which meet one another, are parallel to DC and CF which meet one another and are not in the ſame plane with the other two;
b. 10. 11. wherefor they ſhall contain equal angles[b]; the angle ABH is therefor equal to the angle DCF. and becauſe AB, BH are equal to DC, CF, and the angle ABH equal to the angle DCF, the baſe AH ſhall be
c. 4. 1. equal [c] to the baſe DF, and the triangle ABH to the triangle DCF.
d. 34. 1. and the parallelogram BG is double [d] of the triangle ABH, and the parallelogram CE double of the triangle DCF; therefor the parallelogram BG is equal and ſimilar to the parallelogram CE. in the ſame manner, it ſhall be proved that the parallelogram AC is equal and ſimilar



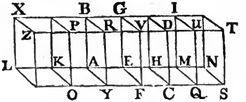
milar to the parallelogram GF, and the parallelogram AE to BF. Therefor if a ſolid &c. Q. E. D.

PROP. XXV. THEOR.

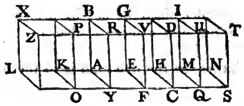
IF a ſolid parallelepiped be cut by a plane parallel to two of its oppoſite planes; it ſhall divide the whole into two ſolids, the baſe of one of which ſhall be to the baſe of the other, as the one ſolid is to the other.

Let the ſolid parallelepiped ABCD be cut by the plane EV which is parallel to the oppoſite planes AR, HD, and divides the whole into the two ſolids ABFV, EGCD; as the baſe AEFY of the firſt is to the baſe EHCF of the other, ſo ſhall the ſolid ABFV be to the ſolid EGCD.

Produce AH both ways, and take any number of ſtraight lines HM, MN each equal to EH, and any number AK, KL each equal to EA, and compleat the parallelograms LO, KY, HQ, MS, and the ſolids LP, KR, HU, MT. therefor becauſe the ſtraight lines LK, KA, AE are all equal, the parallelograms LO, KY, AF ſhall be equal[a]. a. 36. 1.
and likewiſe the parallelograms KX, KB, AG; as alſo [b] the parallelograms LZ, KP, AR, becauſe they are oppoſite planes. by the ſame reaſon, the parallelograms EC, HQ, MS are equal[a]; and the paral- b. 24. 11.



lelograms HG, HI, IN, as alſo [b] HD, MU, NT. therefor three planes b. 24. 11. of the ſolid LP, are equal and ſimilar to three planes of the ſolid KR, as alſo to three planes of the ſolid AV. but the three planes oppoſite



to theſe three are equal and ſimilar [b] to them in the ſeveral ſolids, and none of their ſolid angles are contained by more than three plane c. C. 11. angles. therefor the three ſolids LP, KR, AV are equal [c] to one another. by the ſame reaſon, the three ſolids ED, HU, MT are equal to one another. therefor what multiple ſoever the baſe LF is of the baſe AF, the ſame multiple is the ſolid LV of the ſolid AV. by the ſame reaſon, whatever multiple the baſe NF is of the baſe HF, the ſame multiple is the ſolid NV of the ſolid ED. and if the baſe LF be equal to the baſe NF, the ſolid LV is equal [c] to the ſolid NV; and if the baſe LF be greater than the baſe NF, the ſolid LV is greater than the ſolid NV; and if leſs, leſſer. ſince then there are four magnitudes, viz. the two baſes AF, FH, and the two ſolids AV, ED, and of the baſe AF and ſolid AV, the baſe LF and ſolid LV are any equimultiples whatever; and of the baſe FH and ſolid ED, the baſe FN and ſolid NV are any multiples whatever; and it has been proved, that if the baſe LF is greater than the baſe FN, the ſolid LV is greater than d. 5. Def. 5. the ſolid NV; and if equal, equal; and if leſs, leſſer. Therefor [d] as the baſe AF is to the baſe FH, ſo is the ſolid AV to the ſolid ED. Wherefor if a ſolid &c. Q. E. D.

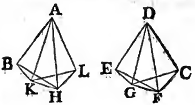
PROP. XXVI.

PROP. XXVI. PROB.

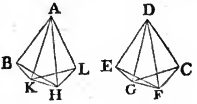
AT a given point in a given ſtraight line, to make a ſolid angle equal to a given ſolid angle contained by three plane angles.

Let AB be a given ſtraight line, A a given point in it, and D a given ſolid angle contained by the three plane angles EDC, EDF, FDC. it is required to make at the point A in the ſtraight line AB a ſolid angle equal to the ſolid angle D.

In the ſtraight line DF take any point F, from which draw [a] FG a. 11. 11. perpendicular to the plane EDC, meeting that plane in G; join DG, and at the point A in the ſtraight line AB make [b] the angle BAL b. 23. 1. equal to the angle EDC, and in the plane BAL make the angle BAK equal to the angle EDG; then make AK equal to DG, and from the point K erect [c] KH at c. 12. 11. right angles to the plane BAL; and make KH equal to GF. the ſolid angle at A which is contained by the three plane angles BAL, BAH, HAL ſhall be equal to the ſolid angle at D contained by the three plane angles EDC, EDF, FDC.



Take the equal ſtraight lines AB, DE, and join HB, KB, FE, GE. and becauſe FG is perpendicular to the plane EDC, it ſhall make right angles [d] with every ſtraight line in that plane which meets it. d. 3. Def. 11. therefor each of the angles FGD, FGE is a right angle. by the ſame reaſon, HKA, HKB are right angles. and becauſe KA, AB are equal to GD, DE, each to each, and contain equal angles, the baſe BK ſhall be equal [e] to the baſe EG. and KH is equal to GF, and HKB, e. 4. 1. FGE are right angles, therefor HB is equal [e] to FE. again, becauſe AK, KH are equal to DG, GF, and contain right angles, the baſe AH ſhall be equal to the baſe DF; and AB is equal to DE; therefor HA, AB are equal to FD, DE, and the baſe HB is equal to the baſe FE; f. 8. 1. therefor the angle BAH is equal [f] to the angle EDF. by the ſame reaſon, the angle HAL is equal to the angle FDC. becauſe if AL and DC be made equal, and KL, HL, GC, FC be joined, ſince the whole angle BAL is equal to the whole EDC, and the parts of them BAK, EDG are, by the conſtruction, equal, the remaining angle KAL ſhall be equal to the remaining angle GDC. and becauſe KA, AL are equal to GD, DC, and contain equal angles, the baſe KL ſhall be equal [e] to the baſe GC. and KH is equal to GF, ſo that LK, KH are equal to CG, GF, and they contain right angles; therefor the baſe HL is equal to the baſe FC. again, becauſe HA, AL are equal to FD, DC, and the baſe HL to the baſe FC, the angle HAL ſhall be equal [f] to the angle FDC. therefor becauſe the three plane angles BAL, BAH, HAL which contain the ſolid angle at A, are equal to the three plane angles EDC, EDF, FDC which contain the ſolid angle at D, each to each, and are ſituated in the ſame order; the ſolid angle at A ſhall be g. B. 11. equal [g] to the ſolid angle at D. Therefor at a given point in a given ſtraight line a ſolid angle has been made equal to a given ſolid angle contained by three plane angles. Which was to be done.

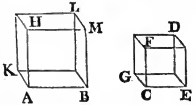


PROP. XXVII.

PROP. XXVII. PROB.

TO deſcribe from a given ſtraight line a ſolid parallelepiped ſimilar, and ſimilarly ſituated to one given.

Let AB be the given ſtraight line, and CD the given ſolid parallelepiped. It is required from AB to deſcribe a ſolid parallelepiped ſimilar, and ſimilarly ſituated to CD.



At the point A of the given ſtraight line AB make [a] a ſolid angle equal to the ſolid angle at C, and let BAK, KAH, HAB be the three plane angles which contain it, ſo that BAK be equal to the angle ECG, and KAH to GCF, and HAB to FCE. and as EC to CG, ſo make [b] BA to AK, and as GC to CF, ſo make [b] KA to AH; wherefor ex aequali [c], as EC to CF, ſo ſhall BA be to AH. compleat the parallelogram BH, and the ſolid AL. and becauſe, as EC to CG, ſo BA to AK, the ſides about the equal angles ECG, BAK are proportionals; therefor the parallelogram BK is ſimilar to EG. by the ſame reaſon the parallelogram KH is ſimilar to GF, and HB to FE. wherefor three parallelograms of the ſolid AL are ſimilar to three of the ſolid CD; and the three oppoſite ones in each ſolid are equal [d] and ſimilar to theſe, each to each. alſo, becauſe the plane angles which contain the ſolid angles of the figures are equal, each to each, and ſituated in the ſame order, the ſolid angles ſhall be equal [e], each to each. Therefor the ſolid AL is ſimilar [f] to the ſolid CD. wherefor from a given ſtraight line AB a ſolid parallelepiped

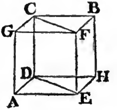
a. 26. 11.
b. 12. 6.
c. 22. 5.
d. 24. 11.
e. B. 11.
f. 11. Def. 11.

parallelepiped AL has been deſcribed ſimilar, and ſimilarly ſituated to the given one CD. Which was to be done.

PROP. XXVIII. THEOR.

IF a ſolid parallelepiped be cut by a plane paſſing through the diagonals of two of the oppoſite planes; it ſhall be cut into two equal parts.

Let AB be a ſolid parallelepiped, and DE, CF the diagonals of
the oppoſite parallelograms AH, GB, viz. thoſe which are drawn betwixt
the equal angles in each. and becauſe CD, FE are each of them
parallel to GA, and not in the ſame plane with it, CD, EF ſhall be
a. 9. 11. parallel[a]; wherefor the diagonals CF, DE are in the plane in which
the parallels are, and ſhall themſelves be parallel[b].
b. 16. 11. and the plane CDEF ſhall cut the ſolid
AB into two equal parts.



c. 34. 1. Becauſe the triangle CGF is equal [c] to the triangle
CBF, and the triangle DAE to DHE; and
d. 24. 11. that the parallelogram CA is equal [d] and ſimilar to
the oppoſite one BE; and the parallelogram GE
to CH: the priſm contained by the two triangles CGF, DAE, and
e. C. 11. the three parallelograms CA, GE, EC, ſhall be equal [e] to the priſm
contained by the two triangles CBF, DHE, and the three parallelograms
BE, CH, EC; becauſe they are contained by the ſame number
of equal and ſimilar planes, alike ſituated, and none of their ſolid angles
are contained by more than three plane angles. Therefor the ſolid
AB is cut into two equal parts by the plane CDEF. Q. E. D.

N. B. The

'N. B. The inſiſting ſtraight lines of a parallelepiped mentioned 'in the next and ſome following Propoſitions, are the ſides of the pa'rallelograms betwixt the baſe and the oppoſite plane parallel to it.'

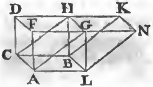
PROP. XXIX. THEOR.

SOLID parallelepipeds upon the ſame baſe, and of the ſame altitude, whoſe inſiſting ſtraight lines are terminated in the ſame ſtraight lines in the plane oppoſite to the baſe, are equal to one another.

See the figures in the following page.

Let the ſolid parallelepipeds AH, AK be upon the ſame baſe AB, and of the ſame altitude, and their inſiſting ſtraight lines AF, AG, LM, LN; CD, CE, BH, BK be terminated in the ſame ſtraight lines FN, DK. the ſolid AH ſhall be equal to the ſolid AK.

Firſt, Let the parallelograms DG, HN which are oppoſite to the baſe AB have a common ſide HG. becauſe therefor the ſolid AH is cut by the plane AGHC paſſing through the diagonals AG, CH of the oppoſite planes ALGF, CBHD, AH ſhall be cut into two equal parts[a] by the plane AGHC. therefor the ſolid AH is double of the priſm which is contained by the triangles ALG, CBH. by the ſame reaſon, becauſe the ſolid AK is cut by the plane LGHB through the diagonals LG, BH of the oppoſite planes ALNG, CBKH, the ſolid AK is double of the ſame priſm which is contained by the triangles ALG, CBH. Therefor the ſolid AH is equal to the ſolid AK.



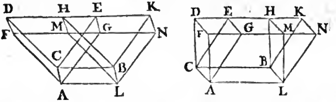
a. 28. 11.

But let the parallelograms DM, EN oppoſite to the baſe have no common ſide. then becauſe CH, CK are parallelograms, CB ſhall be equal [b] to each of the oppoſite ſides DH, EK; wherefor DH is equal

b. 34. 1.

to

Book XI. to EK. add, or take away the common part HE; DE ſhall be equal c. 38. 1. to HK. wherefor alſo the triangle CDE is equal [c] to the triangle d. 36. 1. BHK. and the parallelogram DG is equal [d] to the parallelogram HN. by the ſame reaſon, the triangle AFG is equal to the triangle e. 24. 11. LMN. and the parallelogram CF is equal [e] to the parallelogram BM, and CG to BN; for they are oppoſite. Therefor the priſm which is contained by the two triangles AFG, CDE, and the three parallelograms



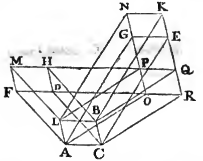
f. C. 11. AD, DG, GC is equal [f] to the priſm contained by the two triangles LMN, BHK, and the three parallelograms BM, MK, KL. If therefor the priſm LMN, BHK be taken from the ſolid whoſe baſe is the parallelogram AB, and FDKN the one oppoſite to it; and if from this ſame ſolid there be taken the priſm AFG, CDE; the remaining ſolid, viz. the parallelepiped AH, ſhall be equal to the remaining parallelepiped AK. Therefor ſolid parallelepipeds &c. Q. E. D.

PROP. XXX. THEOR.

SOLID parallelepipeds upon the ſame baſe, and of the ſame altitude, whoſe inſiſting ſtraight lines are not terminated in the ſame ſtraight lines in the plane oppoſite to the baſe, are equal to one another.

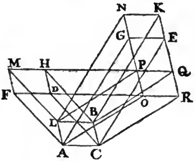
Let the parallelepipeds CM, CN be upon the ſame baſe AB, and of the ſame altitude, but their inſiſting ſtraight lines AF, AG, LM, LN, CD, CE, BH, BK not terminated in the ſame ſtraight lines. the ſolids CM, CN ſhall be equal to one another.

Produce FD, MH, and NG, KE, and let them meet one another in the points O, P, Q, R; and join AO, LP, BQ, CR. and becauſe the plane LBHM is parallel to the oppoſite plane ACDF, and that the plane LBHM is that in which are the parallels LB, MHPQ, in which alſo is the figure BLPQ; and the plane ACDF is that in which are the parallels AC, FDOR, in which alſo is the figure CAOR; therefor the figures BLPQ, CAOR are in parallel planes. in like manner, becauſe the plane ALNG is parallel to the oppoſite plane CBKE, and that the plane ALNG is that in which are the parallels AL, OPGN, in which alſo is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, in which alſo is the figure CBQR; therefor the figures ALPO, CBQR are in parallel planes. and the planes ACBL, ORQP are parallel; therefor the ſolid CP is a parallelepiped. but the ſolid CM whoſe baſe is ACBL, and FDHM the parallelogram oppoſite to it, is equal [a] to the ſolid CP whoſe baſe is the parallelogram ACBL, and ORQP the one oppoſite to it; becauſe they are upon the ſame baſe, and their inſiſting



a. 29. 11.

ſtraight lines AF, AO, CD, CR; LM, LP, BH, BQ are in the ſame a. 29. 11. ſtraight lines FR, MQ. and the ſolid CP is equal [a] to the ſolid CN,



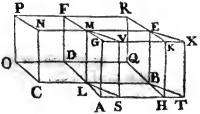
for they are upon the ſame baſe ACBL, and their inſiſting ſtraight lines AO, AG, LP, LN; CR, CE, BQ, BK are in the ſame ſtraight lines ON, RK. therefor the ſolid CM is equal to the ſolid CN. Wherefor ſolid parallelepipeds &c. Q. E. D.

PROP. XXXI. THEOR.

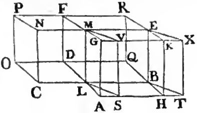
SOLID parallelepipeds which are upon equal baſes, and of the ſame altitude, are equal to one another.

Let the ſolid parallelepipeds AE, CF, be upon equal baſes AB, CD, and of the ſame altitude; the ſolid AE ſhall be equal to the ſolid CF.

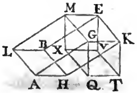
Firſt, Let the inſiſting ſtraight lines be at right angles to the baſes AB, CD, and let the baſes be placed in the ſame plane, and ſo as that the ſides CL, LB be in a ſtraight line; therefor the ſtraight line LM which is at right angles to the plane in which the baſes are, in the a. 13. 11. point L, ſhall be common [a] to the two ſolids AE, CF; let the other inſiſting lines of the ſolids be AG, HK, BE; DF, OP, CN. and firſt, let the angle ALB be equal to the angle CLD; then AL, LD ſhall be in a ſtraight line*. produce OD, HB, and let them meet in Q, * 14. 1. and compleat the ſolid parallelepiped LR whoſe baſe is the parallelogram LQ, and LM one of its inſiſting ſtraight lines. therefor becauſe the parallelogram AB is equal to CD, as the baſe AB is to the baſe LQ, ſo ſhall [b] the baſe CD be to the ſame LQ. and becauſe the b. 7. 5. ſolid parallelepiped AR is cut by the plane LMEB which is parallel to the oppoſite planes AK, DR; as the baſe AB is to the baſe LQ, ſo ſhall [c] the ſolid AE be to the ſolid LR. c. 25. 11. by the ſame reaſon, becauſe the ſolid parallelepiped CR is cut by the plane LMFD which is parallel to the oppoſite planes CP, BR; as the baſe CD to the baſe LQ, ſo is the ſolid CF to the ſolid LR. but as the baſe AB to the baſe LQ, ſo the baſe CD to the baſe LQ, as before was proved. therefor as the ſolid AE to the ſolid LR, ſo is the ſolid CF to the ſolid LR; and therefor the ſolid AE is equal [d] to the ſolid CF. d. 9. 5.

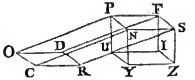


But let the ſolid parallelepipeds SE, CF be upon equal baſes SB, CD, and of the ſame altitude, and let their inſiſting ſtraight lines be at right angles to the baſes; and place the baſes SB, CD in the ſame plane, ſo that CL, LB be in a ſtraight line; and let the angles SLB, CLD be unequal; the ſolid SE ſhall be equal to the ſolid CF. produce DL, TS until they meet in A, and from B draw BH parallel to DA; and let HB, OD produced meet in Q, and compleat the ſolids AE, LR. therefor the ſolid AE whoſe baſe is the parallelogram LE, and the oppoſite to it AK, is equal [e] to the ſolid SE whoſe baſe is e. 29. 11. LE, and SX oppoſite to it; for they are upon the ſame baſe LE, and of the ſame altitude, and their inſiſting ſtraight lines, viz. LA, LS, BH, BT; MG, MV, EK, EX are in the ſame ſtraight lines AT, GX. and becauſe the parallelogram f. 35. 1. AB is equal [f] to SB, for they are upon the ſame baſe LB, and between the ſame parallels LB, AT; and that the baſe SB is equal to the baſe CD; therefor the baſe AB ſhall be equal to the baſe CD. and the angle ALB is equal to the angle CLD, therefor, by the firſt caſe, the ſolid AE is equal to the ſolid CF; but the ſolid AE is equal to the ſolid SE, as was demonſtrated; therefor the ſolid SE is equal to the ſolid CF.



But if the inſiſting ſtraight lines AG, HK, BE, LM; CN, RS, DF, OP, be not at right angles to the baſes AB, CD; likewiſe in this caſe the ſolid AE ſhall be equal to the ſolid CF. from the points G, K, E, M; N, S, F, P, draw the ſtraight lines GQ, g. 11. 11. KT, EV, MX; NY, SZ, FI, PU, perpendicular [g] to the plane in which are the baſes AB, CD; and let them meet it in the points Q, T, V, X; Y, Z, I, U, and join QT, TV, VX, XQ; YZ, ZI, IU, UY. therefor becauſe GQ, KT are at right angles to the ſame h. 6. 11. plane, they ſhall be parallel [h] to one another. and MG, EK are parallel; therefor the planes MQ, ET of which one paſſes through MG,





MG, GQ, and the other through EK, KT which are parallel to MG, GQ, and not in the ſame plane with them, ſhall be parallel [i] to one another. by the ſame reaſon, the planes MV, GT are parallel to one another. therefor the ſolid QE is a parallelepiped. in like manner, it ſhall be proved that the ſolid YF is a parallelepiped. but, from what has been demonſtrated, the ſolid EQ is equal to the ſolid FY, becauſe they are upon equal baſes MK, PS, and of the ſame altitude, and have their inſiſting ſtraight lines at right angles to the baſes. and the ſolid EQ is equal [k] to the ſolid AE; and the ſolid FY to the ſolid CF; becauſe they are upon the ſame baſes and of the ſame altitude. therefor the ſolid AE is equal to the ſolid CF. Wherefor ſolid parallelepipeds &c. Q. E. D.

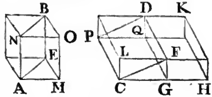
i. 15. 11.

k. 29. or 30. 11.

PROP. XXXII. THEOR.

SOLID parallelepipeds which have the ſame altitude, are to one another as their baſes.

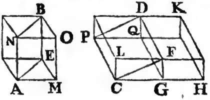
Let AB, CD be ſolid parallelepipeds of the ſame altitude. they ſhall be to one another as their baſes; that is, as the baſe AE to the baſe CF, ſo the ſolid AB to the ſolid CD.



To the ſtraight line FG apply the parallelogram FH equal [a] to AE, ſo that the angle FGH be equal to the angle LCG; and compleat the ſolid parallelepiped GK upon the baſe FH, and one of whoſe inſiſting lines is FD, whereby they ſhall be of the ſame altitude. therefor the ſolid AB is equal [b] to the ſolid GK, becauſe they are upon equal baſes AE, FH, and of the ſame altitude. and becauſe the ſolid parallelepiped CK is cut by the plane

a. Cor. 45. 1.

b. 31. 11.



plane DG which is parallel to its oppoſite planes, the baſe HF ſhall [c]
c. 25. 11. be to the baſe FC, as the ſolid HD to the ſolid DC. but the baſe HF is equal to the baſe AE, and the ſolid GK to the ſolid AB. therefor as the baſe AE to the baſe CF, ſo is the ſolid AB to the ſolid CD. Wherefor ſolid parallelepipeds &c. Q. E. D.

Cor. From this it is manifeſt, that priſms upon triangular baſes, of the ſame altitude, are to one another as their baſes.

Let the priſms whoſe baſes are the triangles AEM, CFG, and NBO, PDQ the triangles oppoſite to them, have the ſame altitude; and compleat the parallelograms AE, CF, and the ſolid parallelepipeds AB, CD, in the firſt of which let MO be one of the inſiſting lines, and GQ in the other. and becauſe the ſolid parallelepipeds AB, CD have the ſame altitude, they ſhall be to one another as the baſe AE
d. 28. 11. is to the baſe CF; wherefor the priſms, which are their halfs, [d] ſhall be to one another as the baſe AE to the baſe CF; that is, as the triangle AEM to the triangle CFG.

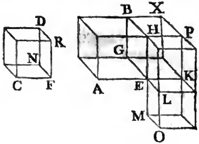
PROP. XXXIII. THEOR.

SIMILAR ſolid parallelepipeds are one to another in the triplicate ratio of their homologous ſides.

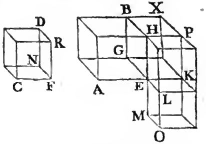
Let AB, CD be ſimilar ſolid parallelepipeds, and the ſide AE homologous to the ſide CF. the ſolid AB ſhall have to the ſolid CD, the triplicate ratio of that which AE has to CF.

Produce AE, GE, HE, and in theſe produced take EK equal to CF, EL equal to FN, and EM equal to FR; and compleat the parallelogram KL, and the ſolid KO. becauſe KE, EL are equal to CF, FN, and the angle KEL equal to the angle CFN, becauſe the angle AEG is equal to CFN, by reaſon the ſolids AB, CD are ſimilar; the parallelogram KL ſhall be ſimilar and equal to the parallelogram CN. by the ſame reaſon, the parallelogram MK is ſimilar and equal to CR, and alſo OE to FD. therefor three parallelograms of the ſolid KO are equal and ſimilar to three parallelograms of the ſolid CD. and the three oppoſite ones in each ſolid are equal [a] and ſimilar to theſe. a. 24. 11. therefor the ſolid KO is equal [b] b. C. 11. and ſimilar to the ſolid CD. compleat the parallelogram GK, and compleat the ſolids EX, LP upon the baſes GK, KL, ſo that EH be an inſiſting ſtraight line in each of them, whereby they ſhall be of the ſame altitude with the ſolid AB. and becauſe the ſolids AB, CD are ſimilar, and by permutation, as AE is to CF, ſo is EG to FN, and ſo is EH to FR; and FC is equal to EK, and FN to EL, and FR to EM; therefor as AE to EK, ſo is EG to EL, and ſo is HE to EM. but as AE to EK, ſo [c] is the parallelogram c. 1. 6. AG to the parallelogram GK; and as GE to EL, ſo [c] is GK to KL; and as HE to EM, ſo [c] is PE to KM. therefor as the parallelogram AG to the parallelogram GK, ſo is GK to KL, and PE to KM. but as AG to GK, ſo [d] is the ſolid AB to the ſolid EX; and as GK to KL, d. 25. 11. ſo [d] is the ſolid EX to the ſolid PL; and as PE to KM, ſo [d] is the ſolid PL to the ſolid KO. and therefor as the ſolid AB to the ſolid

EX,



EX, ſo is EX to PL, and PL to KO. but if four magnitudes be continual proportionals, the firſt is ſaid to have to the fourth the triplicate ratio of that which it has to the ſecond. therefor the ſolid AB has to the ſolid KO, the triplicate ratio of that which AB has to EX. but as AB is to EX, ſo is the parallelogram AG to the parallelogram GK, and the ſtraight line AE to the ſtraight line EK. wherefor the ſolid AB has to the ſolid KO, the triplicate ratio of that which AE has to EK. and the ſolid KO is equal to the ſolid CD, and the ſtraight line EK is equal to the ſtraight line CF. Therefor the ſolid AB has to the ſolid CD, the triplicate ratio of that which the ſide AE has to the homologous ſide CF. Q. E. D.



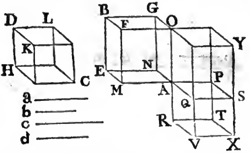
Cor. From this it is manifeſt, if four ſtraight lines be continual proportionals, as the firſt is to the fourth, ſo is the ſolid parallelepiped deſcribed from the firſt to the ſimilar ſolid ſimilarly deſcribed from the ſecond; becauſe the firſt ſtraight line has to the fourth, the triplicate ratio of that which it has to the ſecond.

PROP. D. THEOR.

SOLID parallelepipeds which are contained by parallelograms that are equiangular to one another, each to each, that is, whoſe ſolid angles are equal, each to each; have to one another the ratio which is the ſame with the ratio compounded of the ratios of their ſides.

Let AB, CD be ſolid parallelepipeds, of which AB is contained by the parallelograms AE, AF, AG which are equiangular, each to each, to the parallelograms CH, CK, CL which contain the ſolid CD. the ratio which the ſolid AB has to the ſolid CD ſhall be the ſame with that which is compounded of the ratios of the ſides AM to DL, AN to DK, and AO to DH.

Produce MA, NA, OA to P, Q, R, ſo that AP be equal to DL, AQ to DK, and AR to DH; and compleat the ſolid parallelepiped

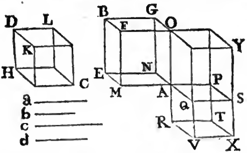


AX contained by the parallelograms AS, AT, AV ſimilar and equal to CH, CK, CL, each to each. therefor the ſolid AX is equal [a] to the ſolid CD. compleat likewiſe the ſolid AY whoſe baſe is AS, and AO one of its inſiſting ſtraight lines. Take any ſtraight line a, and as MA to AP, ſo make a to b; and as NA to AQ, ſo make b to c; and as OA to AR, ſo c to d. therefor becauſe the parallelogram AE is equiangular to AS, AE ſhall be to AS, as the ſtraight line a to c, as is demonſtrated in the 23. Prop. Book 6. and the ſolids AB, AY, being betwixt the parallel planes BOY, EAS, are of the ſame altitude. therefor the ſolid AB is to the ſolid AY, as [b] the baſe AE to the baſe AS; that is, as the ſtraight line a is to c. and the ſolid AY is to the ſolid AX, as [c] the baſe OQ is to the baſe QR; that is, as the ſtraight line

a. C. 11.
b. 32. 11.
c. 25. 11.

OA to AR; that is, as the ſtraight line c to the ſtraight line d. therefor becauſe the ſolid AB is to the ſolid AY, as a is to c, and the ſolid AY to the ſolid AX, as c is to d; ex aequali, the ſolid AB ſhall be to the ſolid AX, or CD which is equal to it, as the ſtraight line a is to d. but the ratio of a to d is ſaid to be compounded [d] of the ratios of a to b, b to c, and c to d, which are the ſame with the ratios of the ſides MA to AP, NA to AQ, and OA to AR, each to each. and the ſides AP, AQ, AR are equal to the ſides DL, DK, DH, each to each. Therefor the ſolid AB has to the ſolid CD the ratio which is the ſame with that which is compounded of the ratios of the ſides AM to DL, AN to DK, and AO to DH. Q. E. D.

d. Def. A. 5.

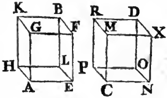


PROP. XXXIV. THEOR.

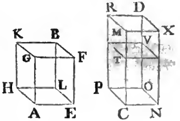
THE baſes and altitudes of equal ſolid parallelepipeds, are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the ſolid parallelepipeds are equal.

Let AB, CD be equal ſolid parallelepipeds; their baſes ſhall be reciprocally proportional to their altitudes; that is, as the baſe EH is to the baſe NP, ſo ſhall the altitude of the ſolid CD be to the altitude of the ſolid AB.

Firſt, Let the inſiſting ſtraight lines AG, EF, LB, HK; CM, NX, OD, PR be at right angles to the baſes. as the baſe EH to the baſe NP, ſo ſhall CM be to AG. if the baſe EH be equal to the baſe NP, then becauſe the ſolid AB is likewiſe equal to the ſolid CD, CM ſhall be equal to AG. becauſe if the baſes EH, NP be equal, but the altitudes AG, CM be not equal, neither ſhall the ſolid AB be equal to the ſolid CD. but the ſolids are equal, by the Hypotheſis. therefor the altitude CM is not unequal to the altitude AG; that is, they are equal. wherefor as the baſe EH to the baſe NP, ſo is CM to AG.



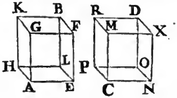
Next, let the baſes EH, NP not be equal, but EH greater than the other. ſince then the ſolid AB is equal to the ſolid CD, CM ſhall be greater than AG. for if it be not, neither alſo, in this caſe, ſhould the ſolids AB, CD be equal, which, by the Hypotheſis, are equal. Make then CT equal to AG, and compleat the ſolid parallelepiped CV whoſe baſe is NP, and altitude CT. Becauſe the ſolid AB is equal to the ſolid CD, the ſolid AB ſhall be to the ſolid CV, as [a] the ſolid CD to the ſolid CV. but as the ſolid AB to the ſolid CV, ſo [b] is the baſe EH to the baſe NP; for the ſolids AB, CV are of the ſame altitude; and as the ſolid CD to CV, ſo [c] is the baſe MP to the baſe PT, and ſo [d] is the ſtraight line MC to CT; and CT is equal to AG. therefor as the



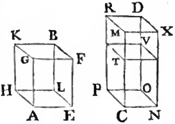
a. 7. 5.
b. 32. 11.
c. 25. 11.
d. 1. 6.

baſe EH to the baſe NP, ſo is MC to AG. wherefor the baſes of the ſolid parallelepipeds AB, CD are reciprocally proportional to their altitudes.

Let now the baſes of the ſolid parallelepipeds AB, CD be reciprocally proportional to their altitudes; viz. as the baſe EH to the baſe NP, ſo the altitude of the ſolid CD to the altitude of the ſolid AB; the ſolid AB ſhall be equal to the ſolid CD. let the inſiſting lines be, as before, at right angles to the baſes: then, if the baſe EH be equal to the baſe NP, ſince EH is to NP, as the altitude of the ſolid CD is to the altitude of the ſolid AB, the altitude of CD ſhall be e. A. 5. equal [e] to the altitude of AB. but ſolid parallelepipeds upon equal f. 31. 11. baſes, and of the ſame altitude are equal [f] to one another; therefor the ſolid AB is equal to the ſolid CD.

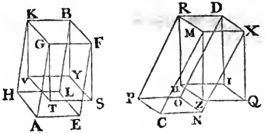


But let the baſes EH, NP be unequal, and let EH be the greater of the two. therefor, ſince as the baſe EH to the baſe NP, ſo is CM the altitude of the ſolid CD to AG the altitude of AB, CM ſhall be greater [e] than AG. take again, CT equal to AG, and compleat, as before, the ſolid CV. and, becauſe the baſe EH is to the baſe NP, as CM to AG, and that AG is equal to CT, the baſe EH ſhall be to the baſe NP, as MC to CT. b. 32. 11. but as the baſe EH is to NP, ſo [b] is the ſolid AB to the ſolid CV; for the ſolids AB, CV are of the ſame altitude; and as MC to CT, ſo

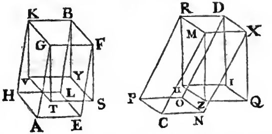


ſo is the baſe MP to the baſe PT, and the ſolid CD to the ſolid [c] CV. and therefor as the ſolid AB to the ſolid CV, ſo is the ſolid CD to the ſolid CV; that is, each of the ſolids AB, CD has the ſame ratio to the ſolid CV; and therefor the ſolid AB is equal to the ſolid CD. c. 25. 11.

Second general Caſe. Let the inſiſting ſtraight lines FE, BL, GA, KH; XN, DO, MC, RP not be at right angles to the baſes of the ſolids; and from the points F, B, K, G; X, D, R, M draw perpendiculars to the planes in which are the baſes EH, NP, meeting theſe planes in the points S, Y, V, T; Q, I, U, Z; and compleat the ſolids FV, XU, which ſhall be parallelepipeds, as was proved in the laſt part of Prop. 31ſt of this Book. In this caſe likewiſe, if the ſolids AB, CD be equal, their baſes ſhall be reciprocally proportional to their altitudes, viz. the baſe EH to the baſe NP, as the altitude of the ſolid CD to the altitude of the ſolid AB. Becauſe the ſolid AB is equal to the ſolid CD, and that the ſolid BT is equal [g] to the ſolid BA, for they are upon the ſame baſe FK, and of the ſame altitude; and the ſolid DC equal [g] to the ſolid DZ, being upon the ſame baſe XR, and of the ſame altitude; the ſolid BT ſhall be equal to the ſolid DZ. but the baſes are reciprocally proportional to the altitudes of equal ſolid parallelepipeds whoſe inſiſting ſtraight lines are at right angles to their baſes, as before was proved. therefor as the baſe FK to the baſe XR, g. 29. or 30. 11.



ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT. and the baſe FK is equal to the baſe EH, and the baſe XR to the baſe NP. wherefor as the baſe EH to the baſe NP, ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT. but the altitudes of the ſolids DZ, DC, as alſo of the ſolids BT, BA are the ſame. Therefor as the baſe EH to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB; that is, the baſes of the ſolid parallelepipeds AB, CD are reciprocally proportional to their altitudes.



Next, Let the baſes of the ſolids AB, CD be reciprocally proportional to their altitudes, viz. the baſe EH to the baſe NP, as the altitude of the ſolid CD to the altitude of the ſolid AB; the ſolid AB ſhall be equal to the ſolid CD. the ſame conſtruction being made, becauſe as the baſe EH to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB; and that the baſe EH is equal to the baſe FK; and NP to XR; the baſe FK ſhall be to the baſe XR, as the altitude of the ſolid CD to the altitude of AB. but the altitudes of the ſolids AB, BT are the ſame, as alſo of CD and DZ; therefor as the baſe FK to the baſe XR, ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT. wherefor the baſes of the ſolids BT, DZ are reciprocally proportional to their altitudes; and their inſiſting ſtraight lines are at right angles to the baſes; wherefor, as was before proved, the ſolid BT is equal to the ſolid DZ. but BT is equal [g] to the ſolid BA, and DZ to the ſolid DC, becauſe they are upon the ſame baſes, and of the ſame altitude. Therefor the ſolid AB is equal to the ſolid CD. Q. E. D.

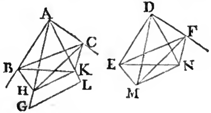
g. 29. or 30. 11.

PROP. XXXV. THEOR.

IF from the vertices of two equal plane angles there be drawn two ſtraight lines elevated above the planes in which are the angles, that contain equal angles with the ſides of theſe angles, each to each; and if in the lines above the planes there be taken any points, and from them perpendiculars be drawn to the planes in which the firſt named angles are; and from the points in which they meet the planes, ſtraight lines be drawn to the vertices of the angles firſt named; theſe ſtraight lines ſhall contain equal angles with the ſtraight lines which are above the planes of the angles.

Let BAC, EDF be two equal plane angles; and from the points A, D let the ſtraight lines AG, DM be elevated above the planes of the angles, making equal angles with their ſides, each to each; viz. the angle GAB equal to the angle MDE, and GAC to MDF; and in AG, DM let any points G, M be taken, and from them be drawn perpendiculars GL, MN to the planes BAC, EDF meeting theſe planes in the points L, N; and join LA, ND. the angle GAL ſhall be equal to the angle MDN.

Make AH equal to DM, and through H draw HK parallel to GL. but GL is perpendicular to the plane BAC, wherefor HK is perpendicular [a] to the ſame plane. from the points K, N, to the a. 8. 11. ſtraight lines AB, AC, DE, DF, draw perpendiculars KB, KC, NE, NF; and join HB, BC, ME, EF. Becauſe HK is perpendicular to the plane BAC, the plane HBK which paſſes through HK ſhall be at b. 18. 11. right angles [b] to the plane BAC; and AB is drawn in the plane BAC at right angles to the common ſection BK of the two planes; there- c. 4. Def. 11. for AB is perpendicular [c] to the plane HBK, and ſhall make right d. 3. Def. 11. angles [d] with every ſtraight line in that plane which meets it. but BH in that plane meets it; therefor ABH is a right angle. by the ſame reaſon, DEM is a right angle, and is therefor equal to the angle ABH.



and the angle HAB is equal to the angle MDE. therefor in the two triangles HAB, MDE there are two angles in one equal to two angles in the other, each to each, and one ſide equal to one ſide, oppoſite to one of the equal angles in each, viz. HA equal to DM; e. 26. 1. therefor the remaining ſides ſhall be equal [e], each to each. wherefor AB is equal to DE. In the ſame manner, if HC and MF be joined, it ſhall be demonſtrated that AC is equal to DF. therefor becauſe AB is equal to DE, BA and AC are equal to ED and DF; and the angle BAC is equal to the angle EDF; wherefor the baſe BC is e- f. 4. 1. qual [f] to the baſe EF, and the remaining angles to the remaining angles. the angle ABC is therefor equal to the angle DEF. and the right angle ABK is equal to the right angle DEN, whence the remaining angle CBK is equal to the remaining angle FEN. by the ſame reaſon, the angle BCK is equal to the angle EFN. therefor in the two triangles BCK, EFN there are two angles in one equal to two angles in the other, each to each, and one ſide equal to one ſide adjacent to the equal angles in each, viz. BC equal to EF; the other ſides therefor are equal to the other ſides; BK then is equal to EN. and AB is equal to DE; wherefor AB, BK are equal to DE, EN; and they contain right angles; wherefor the baſe AK is equal to the baſe DN. and ſince AH is equal to DM, the ſquare of AH ſhall be equal to the ſquare of DM. but the ſquares of AK, KH are equal to the ſquare [g] of AH, becauſe AKH is a right angle. and the ſquares of DN, NM are equal to the ſquare of DM, for DNM is a right angle. wherefor the ſquares of AK, KH are equal to the ſquares of DN, NM; and of theſe the ſquare of AK is equal to the ſquare of DN. therefor the remaining ſquare of KH is equal to the remaining ſquare of NM; and the ſtraight line KH to the ſtraight line NM. and becauſe HA, AK are equal to MD, DN, each to each, and the baſe HK to the baſe MN, as has been proved, the angle HAK ſhall be equal [h] to the angle MDN. Q. E. D.

g. 47. 1.

h. 8. 1.

Cor. From this it is manifeſt, that if from the vertices of two equal plane angles there be elevated two equal ſtraight lines containing equal angles with the ſides of the angles, each to each; the perpendiculars drawn from the extremities of them to the planes of the firſt angles ſhall be equal to one another.

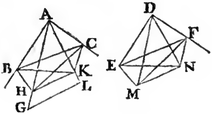
Another Demonſtration of the Corollary.

Let the plane angles BAC, EDF be equal to one another, and let AH, DM be two equal ſtraight lines above the planes of the angles,

containing

containing equal angles with BA, AC, ED, DF, each to each, viz. the angle HAB equal to MDE, and HAC equal to the angle MDF; and from H, M let HK, MN be perpendiculars to the planes BAC, EDF; HK ſhall be equal to MN.

Becauſe the ſolid angle at A is contained by the three plane angles BAC, BAH, HAC, which are, each to each, equal to the three plane angles EDF, EDM, MDF which contain the ſolid angle at D; the ſolid angles at A and D ſhall be equal, and coincide with one another; to wit, if the plane angle ABC be applied to the plane angle EDF, the ſtraight line AH ſhall coincide with DM, as was ſhewn in Prop. B. of this Book. and becauſe AH is equal to DM, the point H ſhall coincide with the point M. wherefor HK which is perpendicular to the plane BAC ſhall coincide with [i] MN which is perpendicular to the plane EDF, becauſe theſe planes coincide with one another. therefor HK is equal to MN. Q. E. D.



i. 13. 11.

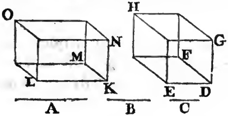
PROP. XXXVI.

PROP. XXXVI. THEOR.

IF three ſtraight lines be proportionals, the ſolid parallelepiped deſcribed from all three as its ſides, is equal to the equilateral parallelepiped deſcribed from the mean proportional, one of whoſe ſolid angles is contained by three plane angles that are equal to the three plane angles that contain one of the ſolid angles of the other figure, each to each.

Let A, B, C be three proportionals, viz. A to B, as B to C. The ſolid deſcribed from A, B, C ſhall be equal to the equilateral ſolid deſcribed from B, which is equiangular to the other.

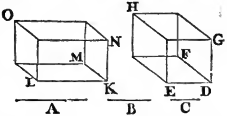
Take a ſolid angle D contained by three plane angles EDF, FDG, GDE; and make each of the ſtraight lines ED, DF, DG equal to B, and compleat the ſolid parallelepiped DH. make LK equal to A, and at the point K in the ſtraight line LK make [a] a ſolid angle contained by the three plane angles LKM, MKN, NKL equal to the angles EDF, FDG, GDE, each to each; and make KN equal to B, and KM equal to C; and compleat the ſolid parallelepiped KO. and becauſe as A is to B, ſo is B to C, and that A is equal to LK, and B to each of the ſtraight lines DE, DF, and C to KM; LK ſhall be to ED, as DF to KM; that is, the ſides about the equal angles are reciprocally proportional; therefor the parallelogram LM is equal [b] to EF. and becauſe EDF, LKM are two equal plane



a. 26. 11.

b. 14. 6.

angles, and the two equal ſtraight lines DG, KN are drawn from their vertices above their planes, and contain equal angles with their ſides; the perpendiculars from the points G, N, to the planes EDF, LKM ſhall be equal e. Cor. 35. 11. [c] to one another. therefor the ſolids KO, DH are of the ſame altitude; and they are upon equal baſes LM, d. 31. 11. EF, and therefor they are equal [d] to one another. but the ſolid KO is deſcribed from the three ſtraight lines A, B, C, and the ſolid DH from the ſtraight line B. If therefor three ſtraight lines &c. Q. E. D.



PROP. XXXVII. THEOR.

If four ſtraight lines be proportionals, the ſimilar ſolid parallelepipeds that are ſimilarly deſcribed from them ſhall alſo be proportionals. and if the ſimilar parallelepipeds ſimilarly deſcribed from four ſtraight lines be proportionals, the ſtraight lines ſhall be proportionals.

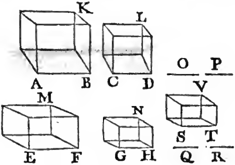
Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. as AB to CD, ſo EF to GH; and let the ſimilar parallelepipeds AK, CL, EM, GN be ſimilarly deſcribed from them. AK ſhall be to CL, as EM to GN.

a. 11. 6. Make [a] AB, CD, O, P continual proportionals, as alſo EF, GH, b. 11. 5. Q, R. and becauſe as AB is to CD, ſo EF to GH; CD ſhall [b] be c. 22. 5. to O, as GH to Q, and O to P, as Q to R; therefor ex aequali [c] d. Cor. 33. 11. AB is to P, as EF to R. but as AB to P, ſo [d] is the ſolid AK to the ſolid CL; and as EF to R, ſo [d] is the ſolid EM to the ſolid GN.

therefor [b] as the ſolid AK to the ſolid CL, ſo is the ſolid EM to the ſolid GN. b. 11. 5.

But let the ſolid AK be to the ſolid CL, as the ſolid EM to the ſolid GN. the ſtraight line AB ſhall be to CD, as EF to GH.

Make as AB to CD, ſo EF to ST, and from ST deſcribe [e] a ſolid parallelepiped SV ſimilar and ſimilarly ſituated to either of the ſolids EM, GN. e. 27. 11. and becauſe AB is to CD, as EF to ST, and that from AB, CD the ſolid parallelepipeds AK, CL are ſimilarly deſcribed;



and in like manner the ſolids EM, SV from the ſtraight lines EF, ST; AK ſhall be to CL, as EM to SV. but, by the Hypotheſis, AK is to CL, as EM to GN. therefor GN is equal [f] to SV. but it is f. 9. 5. likewiſe ſimilar and ſimilarly ſituated to SV; therefor the planes which contain the ſolids GN, SV are ſimilar and equal, and their homologous ſides GH, ST ſhall be equal to one another. and becauſe as AB to CD, ſo EF to ST, and that ST is equal to GH; AB ſhall be to CD, as EF to GH. Therefor if four ſtraight lines &c. Q. E. D.

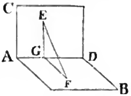
PROP. XXXVIII.

PROP. XXXVIII. THEOR.

“ IF a plane be perpendicular to another plane, and a
“ ſtraight line be drawn from a point in one of the
“ planes perpendicular to the other plane, this ſtraight
“ line ſhall fall on the common ſection of the planes.

“ Let the plane CD be perpendicular to the plane AB, and AD
“ their common ſection; and let any point E be taken in the plane
“ CD; the perpendicular drawn from E to the plane AB ſhall fall
“ on AD.

“ For if it does not, let it, if poſſible, fall off it, as EF; and let it
a. 12. 1. “ meet the plane AB in the point F; and from F draw[a], in the plane
“ AB, a perpendicular FG to DA, which
b. 4. Def. 11. “ ſhall alſo be perpendicular [b] to the plane
“ CD; and join EG. therefor becauſe FG
“ is perpendicular to the plane CD, and the
“ ſtraight line EG in that plane meets it;
c. 3. Def. 11. “ FGE ſhall be a right angle[c]. but EF is
“ alſo at right angles to the plane AB; and therefor EFG is a right
“ angle. wherefor two of the angles of the triangle EFG are equal
“ together to two right angles; which is abſurd. therefor the perpen-
“ dicular from the point E to the plane AB does not fall off the
“ ſtraight line AD. it ſhall therefor fall on it. If therefor a plane &c.
“ Q. E. D.”

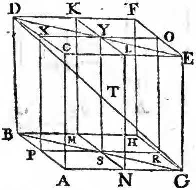


PROP. XXXIX.

PROP. XXXIX. THEOR.

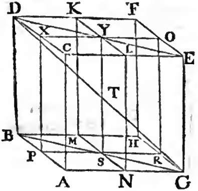
IN a ſolid parallelepiped, if the ſides of two of the oppoſite planes be divided each into two equal parts, the common ſection of the planes paſſing thro' the points of diviſion, and the diameter of the ſolid parallelepiped ſhall cut each other into two equal parts.

Let the ſides of the oppoſite planes CF, AH of the ſolid parallelepiped AF, be divided each into two equal parts in the points K, L, M, N; X, O, P, R; and join KL, MN, XO, PR. and becauſe DK, CL are equal and parallel, KL ſhall be parallel [a] to DC. by the ſame a. 33. 1. reaſon, MN is parallel to BA. and BA is parallel to DC; therefor becauſe KL, BA are each of them parallel to DC, and not in the ſame plane with it, KL ſhall be parallel [b] to BA. and becauſe KL, b. 9. 11. MN are each of them parallel to BA, and not in the ſame plane with it, KL ſhall be parallel [b] to MN; wherefor KL, MN are in one plane. In like manner, it ſhall be proved that XO, PR are in one plane.



plane. Let YS be the common ſection of theſe planes KN, XR; and DG the diameter of the ſolid parallelepiped AF. YS and DG ſhall meet and cut one another into two equal parts.

Join DY, YE, BS, SG. becauſe DX is parallel to OE, the alternate
c. 29. 1. angles DXY, YOE are equal [c] to one another. and becauſe
DX is equal to OE, and XY to YO, and contain equal angles, the
d. 4. 1. baſe DY ſhall be equal [d] to the baſe YE, and the other angles ſhall
be equal; therefor the angle XYD is equal to the angle OYE, and
e. 14. 1. DYE ſhall be a ſtraight [e] line. by the ſame reaſon BSG is a ſtraight



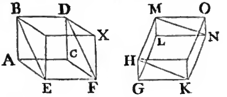
line, and BS equal to SG. and becauſe CA is equal and parallel to DB,
b. 9. 11. and alſo equal and parallel to EG; DB ſhall be equal and parallel [b]
to EG. and DE, BG join their extremities; therefor DE is equal
a. 33. 1. and parallel [a] to BG. and DG, YS are drawn from points in the one
to points in the other, and are therefor in one plane. whence it is
manifeſt that DG, YS muſt meet one another; let them meet in T.
and becauſe DE is parallel to BG, the alternate angles EDT, BGT
f. 15. 1. ſhall be equal [c]; and the angle DTY is equal [f] to the angle GTS.
therefor in the triangles DTY, GTS there are two angles in the one

equal to two angles in the other, and one ſide equal to one ſide, viz. DY to GS; for they are the halfs of DE, BG. therefor the remaining ſides are equal[g], each to each. wherefor DT is equal to TG, and YT equal to TS. Therefor if in a ſolid &c. Q. E. D. [g] 26. 1.

PROP. XL. THEOR.

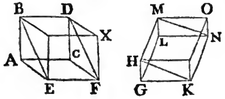
IF there be two triangular priſms of the ſame altitude, the baſe of one of which is a parallelogram, and the baſe of the other a triangle; if the parallelogram be double of the triangle, the priſms ſhall be equal to one another.

Let the priſms ABCDEF, GHKLMN be of the ſame altitude, the firſt whereof is contained by the two triangles ABE, CDF, and the three parallelograms AD, DE, EC; and the other by the two triangles GHK, LMN and the three parallelograms LH, HN, NG; and let one of them have a parallelogram AF, and the other a triangle GHK for its baſe; if the parallelogram AF be double of the triangle GHK, the priſm ABCDEF ſhall be equal to the priſm GHKLMN.



Compleat the ſolids AX, GO; and becauſe the parallelogram AF is double of the triangle GHK; and the parallelogram HK double [a] 34. 1.

of the ſame triangle; the parallelogram AF ſhall be equal to HK. but ſolid parallelepipeds upon equal baſes, and of the ſame altitude
b. 31. 11. are equal [b] to one another. therefor the ſolid AX is equal to the ſolid GO.
c. 28. 11. and the priſm ABCDEF is half [c] of the ſolid AX; and the priſm GHKLMN half [c] of the ſolid GO. therefor the priſm ABCDEF is equal to the priſm GHKLMN. Wherefor if there be two &c. Q. E. D.



THE ELEMENTS OF EUCLID.

BOOK XII.

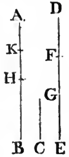
LEMMA I.

Which is the firſt Propoſition of the tenth Book, and is neceſſary to ſome of the Propoſitions of this Book.

IF from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half; and ſo on. there ſhall at length remain a magnitude leſs than the leſſer of the two magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater. if from AB there be taken more than its half, and from the remainder more than its half, and ſo on; there ſhall at length remain a magnitude leſſer than C.

For C may be multiplied ſo as at length to become greater than AB. let it be ſo multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. from AB take BH greater than its half, and from the remainder AH take HK greater than its half, and ſo on until there be as many diviſions in AB as there are in DE. and let the diviſions AK, KH, HB be as many as the diviſions DF, FG, GE. and becauſe DE is greater than AB, and that EG taken from DE is leſſer than its half, but BH taken from AB is greater than its half, the remainder GD ſhall be greater than the remainder HA. again, becauſe GD is greater than HA, and that GF is the half of GD, but HK is greater than the half of HA; the remainder FD ſhall be greater than the remainder AK. and FD is equal to C, therefor C is greater than AK; that is, AK is leſſer than C. Q. E. D.



PROP. I. THEOR.

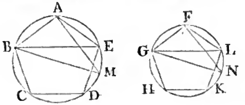
SIMILAR polygons inſcribed in circles, are to one another as the ſquares of their diameters.

Let ABCDE, FGHKL be two circles, and in them the ſimilar polygons ABCDE, FGHKL; and let BM, GN be the diameters of the circles. as the ſquare of BM is to the ſquare of GN, ſo ſhall the polygon ABCDE be to the polygon FGHKL.

Join BE, AM, GL, FN. and becauſe the polygon ABCDE is ſimilar to the polygon FGHKL, the angle BAE ſhall be equal [a] to the angle GFL, and as BA to AE, ſo [a] is GF to FL. wherefor the two triangles BAE, GFL having one angle in one equal to one angle in the

a. 1. Def. 6.

the other, and the ſides about the equal angles proportionals, are e-quiangular[b]; and therefor the angle AEB is equal to the angle FLG. b. 6. 6.
but AEB is equal [c] to the angle AMB, becauſe they ſtand upon the c. 21. 3.
ſame circumference; and the angle FLG is, for the ſame reaſon, e-qual to the angle FNG. therefor alſo the angle AMB is equal to FNG. and the right angle BAM is equal to the right [d] angle GFN; d. 31. 3.
wherefor the remaining angles in the triangles ABM, FGN are equal,



and they are equiangular to one another. therefor as BM to GN, ſo [e] e. 4. 6.
is BA to GF, and the duplicate ratio of BM to GN, ſhall be the ſame [f] to the duplicate ratio of BA to GF. but the ratio of the f. 10. Def. 5. and 22. 5.
ſquare of BM to the ſquare of GN, is the duplicate [g] ratio of that g. 20. 6.
which BM has to GN; and the ratio of the polygon ABCDE to the polygon FGHKL is the duplicate [g] of that which BA has to GF. therefor as the ſquare of BM to the ſquare of GN, ſo is the polygon ABCDE to the polygon FGHKL. wherefor ſimilar polygons &c. Q. E. D.

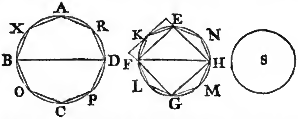
PROP. II. THEOR.

CIRCLES are to one another as the ſquares of their diameters.

Let ABCD, EFGH be two circles, and BD, FH their diameters.

as the ſquare of BD to the ſquare of FH, ſo ſhall the circle ABCD be to the circle EFGH.

For, if it be not ſo, the ſquare of BD ſhall be to the ſquare of FH, as the circle ABCD is to ſome ſpace either leſs than the circle EFGH, or greater than it*. Firſt, let it be to a ſpace S leſs than the circle EFGH; and in the circle EFGH deſcribe the ſquare EFGH. this ſquare is greater than half of the circle EFGH; becauſe if through the points E, F, G, H, there be drawn tangents to the circle, the ſquare
a. 41. 1. EFGH ſhall be half [a] of the ſquare deſcribed about the circle; and



the circle is leſs than the ſquare deſcribed about it; therefor the ſquare EFGH is greater than half of the circle. Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points K, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE. therefor each of the triangles EKF, FLG, GMH, HNE is greater than half of the ſegment of the circle it ſtands in; becauſe if ſtraight lines touching the circle be drawn through the points K, L, M, N, and parallelograms upon the ſtraight lines EF, FG, GH, HE be compleated; each of the triangles EKF, FLG, GMH, HNE ſhall be the half [a] of the parallelogram in which it is. but every ſegment is leſs than the parallelogram in which it is. wherefor each of the triangles EKF, FLG, GMH, HNE is greater than half of the ſegment of the circle which contains it. and if theſe circumferences before named be divided each into two equal parts, and their extremities be joined by ſtraight lines, by continuing to do this, there will at length remain ſegments of the circle which ſhall together be leſs than the exceſs of the circle EFGH above the ſpace S. becauſe, by the preceeding Lemma, if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and ſo on, there ſhall at length remain a magnitude leſs than the leſſer of the two magnitudes. Let then the ſegments EK, KF, FL, LG, GM, MH, HN, NE be thoſe that remain and are together leſs than the exceſs of the circle EFGH above S. therefor the reſt of the circle, viz. the polygon EKFLGMHN ſhall be greater than the ſpace S. Deſcribe likewiſe in the circle ABCD the polygon AXBOCPDR ſimilar to the polygon EKFLGMHN. as, therefor, the ſquare of BD is to the ſquare of FH, ſo [b] is the polygon AXBOCPDR to the polygon EKFLGMHN. b. 1. 12. but the ſquare of BD is alſo to the ſquare of FH, as the circle ABCD is to the ſpace S. therefor as the circle ABCD is to the ſpace S, ſo is [c] the polygon AXBOCPDR to the polygon EKFLGMHN. but c. 11. 5. the circle ABCD is greater than the polygon contained in it; wherefor the ſpace S is greater [d] than the polygon EKFLGMHN. but d. 14. 5. it is likewiſe leſs, as has been demonſtrated; which is impoſſible. Therefor the ſquare of BD is not to the ſquare of FH, as the circle ABCD is to any ſpace leſs than the circle EFGH. In the ſame manner it ſhall be demonſtrated that neither is the ſquare of FH to the ſquare of BD, as the circle EFGH is to any ſpace leſs than the circle ABCD. Nor ſhall the ſquare of BD be to the ſquare of FH, as the circle ABCD is to any ſpace greater than the circle EFGH. for, if poſſible, let it be to T a ſpace greater than the circle EFGH. therefor, inverſely, as the ſquare of FH to the ſquare of BD, ſo is the ſpace T to the circle ABCD. but as the ſpace † T to the circle ABCD, ſo ſhall the circle d. 14. 5. EFGH be to ſome ſpace, which muſt be leſſer [d] than the circle ABCD, becauſe the ſpace T is greater, by Hypotheſis, than the circle EFGH.

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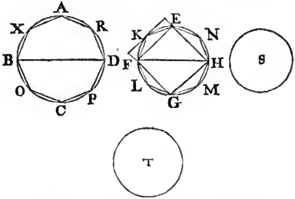
* For there is ſome ſquare equal to the circle ABCD; let P be the ſide of it, and to three ſtraight lines BD, FH and P, there can be a fourth proportional, let this be Q. therefor the ſquares of theſe four ſtraight lines are proportionals; that is, to the ſquares of BD, FH and the circle ABCD it is poſſible there can be a fourth proportional. Let this be S. and in like manner are to be underſtood ſome things in ſome of the following Propoſitions.



therefor as the ſquare of FH is to the ſquare of BD, ſo is the circle EFGH to a ſpace leſſer than the circle ABCD, which has been demonſtrated to be impoſſible. therefor the ſquare of BD is not to the ſquare of FH, as the circle ABCD is to any ſpace greater than the circle EFGH. and it has been demonſtrated that neither is the ſquare of BD to the ſquare of FH, as the circle ABCD to any ſpace leſſer than the circle EFGH. wherefor as the ſquare of BD to the ſquare of FH, ſo is the circle ABCD to the circle EFGH‡. Circles, therefor, are &c. Q. E. D.

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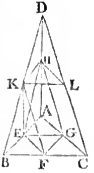
† For as at the foregoing Note at * it was explained how it was poſſible there could be a fourth proportional to the ſquares of BD, FH and the circle ABCD, which was named S. ſo in like manner there can be a fourth proportional to this other ſpace, named T, and the circles ABCD, EFGH. and the like is to be underſtood in ſome of the following Propoſitions.

PROP. III. THEOR.

EVERY pyramid having a triangular baſe, can be divided into two equal and ſimilar pyramids having triangular baſes, and which are ſimilar to the whole pyramid; and into two equal priſms which together are greater than half of the whole pyramid.

Let there be a pyramid whoſe baſe is the triangle ABC and its vertex the point D. the pyramid ABCD can be divided into two equal and ſimilar pyramids having triangular baſes, and ſimilar to the whole; and into two equal priſms which together ſhall be greater than half of the whole pyramid.

Divide AB, BC, CA, AD, DB, DC, each into two equal parts in the points E, F, G, H, K, L, and join EH, EG, GH, HK, KL, LH, EK, KF, FG. Becauſe AE is equal to EB, and AH to HD, HE ſhall be parallel [a] to DB. by the ſame reaſon, HK is parallel to AB. therefor HEBK is a parallelogram, and HK equal [b] to EB. but EB is equal to AE; therefor alſo AE ſhall be equal to HK. and AH is equal to HD; wherefor EA, AH are equal to KH, HD, each to each; and the angle EAH is equal [c] to the angle KHD; therefor the baſe EH is equal to the baſe KD, and the triangle



a. 2. 6.

b. 34. 1.

c. 29. 1.

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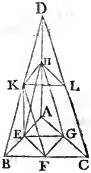
‡ Becauſe as a fourth proportional to the ſquares of BD, FH and the circle ABCD is poſſible, and that it can neither be leſs nor greater than the circle EFGH, it muſt be equal to it.



AEH equal [d] and ſimilar to the triangle HKD. by the ſame reaſon,
d. 4. 1. the triangle AGH is equal and ſimilar to the triangle HLD. and becauſe
the two ſtraight lines EH, HG which meet one another are parallel
to KD, DL that meet one another, and are not in the ſame
e. 10. 11. plane with them, they ſhall contain equal [e] angles; therefor the angle
EHG is equal to the angle KDL. again, becauſe EH, HG are equal
to KD, DL, each to each, and the angle EHG
equal to the angle KDL; the baſe EG ſhall be
equal to the baſe KL, and the triangle EHG
ſhall be equal [d] and ſimilar to the triangle KDL.
by the ſame reaſon, the triangle AEG is alſo
equal and ſimilar to the triangle HKL. Therefor
the pyramid whoſe baſe is the triangle AEG,
f. C. 11. and its vertex the point H, is equal [f] and ſimilar
to the pyramid whoſe baſe is the triangle KHL,
and its vertex the point D. and becauſe HK is
parallel to AB a ſide of the triangle ADB, the
triangle ADB ſhall be equiangular to the triangle HDK, and their
g. 4. 6. ſides ſhall be proportionals [g]. therefor the triangle ADB is ſimilar to
the triangle HDK. and by the ſame reaſon, the triangle DBC is ſimilar
to the triangle DKL; and the triangle ADC to the triangle HDL;
and alſo the triangle ABC to the triangle AEG. but the triangle AEG
is ſimilar to the triangle HKL, as before was proved, therefor the triangle
h. 21. 6. ABC is ſimilar [h] to the triangle HKL. and the pyramid whoſe
baſe is the triangle ABC, and its vertex the point D, is therefor ſimilar
i. B. 11. and 11. Def. 11. [i] to the pyramid whoſe baſe is the triangle HKL, and its vertex
the ſame point D. but the pyramid whoſe baſe is the triangle HKL,
and vertex the point D, is ſimilar, as has been proved, to the pyramid
whoſe baſe is the triangle AEG, and vertex the point H. wherefor

for the pyramid whoſe baſe is the triangle ABC, and vertex the point D, is ſimilar to the pyramid whoſe baſe is the triangle AEG and vertex H. therefor each of the pyramids AEGH, HKLD is ſimilar to the whole pyramid ABCD. and becauſe BF is equal to FC, the parallelogram EBFG ſhall be double [k] of the triangle GFC. but when k. 41. 1.
there are two priſms of the ſame altitude, of which one has a parallelogram for its baſe, and the other a triangle that is half of the parallelogram, theſe priſms are equal [l] to one another; therefor the priſm l. 40. 11.
whoſe baſe is the parallelogram EBFG, and KH the ſtraight line oppoſite to it, is equal to the priſm whoſe baſe is the triangle GFC, and HKL the triangle oppoſite to it; for they are of the ſame altitude, becauſe they are between the parallel [m] planes ABC, HKL. and it m. 15. 11.
is manifeſt that each of theſe priſms is greater than either of the pyramids whoſe baſes are the triangles AEG, HKL, and their vertices the points H, D; becauſe if EF be joined, the priſm whoſe baſe is the parallelogram EBFG, and KH the ſtraight line oppoſite to it, is greater than the pyramid whoſe baſe is the triangle EBF, and vertex the point K; but this pyramid is equal [f] to the pyramid whoſe baſe is the tri- f. C. 11.
angle AEG, and vertex the point H; becauſe they are contained by equal and ſimilar planes. wherefor the priſm whoſe baſe is the parallelogram EBFG, and oppoſite ſide KH, is greater than the pyramid whoſe baſe is the triangle AEG, and vertex the point H. and the priſm whoſe baſe is the parallelogram EBFG, and oppoſite ſide KH is equal to the priſm whoſe baſe is the triangle GFC, and HKL the triangle oppoſite to it; and the pyramid whoſe baſe is the triangle AEG, and vertex H, is equal to the pyramid whoſe baſe is the triangle HKL, and vertex D. therefor the two priſms before mentioned are greater than the two pyramids whoſe baſes are the triangles AEG, HKL, and their vertices the points H, D. Therefor the whole

pyramid whoſe baſe is the triangle ABC, and vertex the point D, is divided into two equal pyramids which are ſimilar to one another, and to the whole pyramid; and into two equal priſms; and the two priſms are together greater than half of the whole pyramid. Q. E. D.

PROP. IV. THEOR.

IF there be two pyramids of the ſame altitude, upon triangular baſes, and each of them be divided into two equal pyramids ſimilar to the whole pyramid, and alſo into two equal priſms; and if each of theſe pyramids be divided in the ſame manner as the firſt two, and ſo on. as the baſe of one of the firſt two pyramids is to the baſe of the other, ſo ſhall all the priſms in one of them be to all the priſms in the other, that are produced by the ſame number of diviſions.

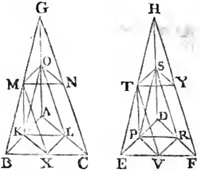
Let there be two pyramids of the ſame altitude upon the triangular baſes ABC, DEF, and having their vertices in the points G, H; and let each of them be divided into two equal pyramids ſimilar to the whole, and into two equal priſms; and let each of the pyramids thus made be conceived to be divided in the like manner, and ſo on. as the baſe ABC is to the baſe DEF, ſo ſhall all the priſms in the pyramid ABCG be to all the priſms in the pyramid DEFH made by the ſame number of diviſions.

Make the ſame conſtruction as in the foregoing propoſition. and a. 2. 6. becauſe BX is equal to XC, and AL to LC, XL ſhall be parallel [a] to AB, and the triangle ABC ſimilar to the triangle LXC. by the ſame reaſon, the triangle DEF is ſimilar to RVF. and becauſe BC is double of CX, and EF double of FV, BC ſhall be to CX, as EF to

FV. and upon BC, CX are deſcribed the ſimilar and ſimilarly ſituated rectilineal figures ABC, LXC; and upon EF, FV, in like manner are deſcribed the ſimilar figures DEF, RVF. therefor as the triangle ABC is to the triangle LXC, ſo [b] is the triangle DEF to the triangle RVF, and, by permutation, as the triangle ABC to the triangle DEF, ſo is the triangle LXC to the triangle RVF. and becauſe the planes ABC, OMN, as alſo the planes DEF, STY are parallel [c], the perpendiculars drawn from the points G, H to the baſes ABC, DEF, which, by the Hypotheſis, are equal to one another, ſhall be cut

b. 22. 6.

c. 15. 11.

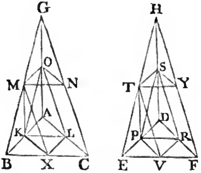


each into two equal [d] parts by the planes OMN, STY, becauſe the ſtraight lines GC, HF are cut into two equal parts in the points N, Y by the ſame planes. therefor the priſms LXCOMN, RVFSTY are of the ſame altitude; and therefor as the baſe LXC to the baſe RVF; that is, as the triangle ABC to the triangle DEF, ſo [e] is the priſm whoſe baſe is the triangle LXC, and the triangle oppoſite to it OMN, to the priſm whoſe baſe is the triangle RVF, and oppoſite triangle STY. and becauſe the two priſms in the pyramid ABCG are equal to one another, and alſo the two priſms in the pyramid DEFH equal

d. 17. 11.

e. Cor. 32. 11.

equal to one another, as the priſm whoſe baſe is the parallelogram KBXL and oppoſite ſide MO, to the priſm whoſe baſe is the triangle LXC and OMN the triangle oppoſite to it; ſo [f] is the priſm whoſe baſe is the parallelogram PEVR, and oppoſite ſide TS, to the priſm whoſe baſe is the triangle RVF, and oppoſite triangle STY. therefor, componendo, as the priſms KBXLMO, LXCOMN together are unto the priſm LXCOMN; ſo are the priſms PEVRTS, RVFSTY to the priſm RVFSTY. and, permutando, as the priſms KBXLMO, LXCOMN are to the priſms PEVRTS, RVFSTY; ſo is the priſm LXCOMN to the priſm RVFSTY. but as the priſm LXCOMN to the priſm RVFSTY, ſo is, as has been proved, the baſe ABC to the baſe DEF. therefor as the baſe ABC to the baſe DEF, ſo are the two priſms in the pyramid ABCG to the two priſms in the pyramid DEFH. and likewiſe if the pyramids now made, for example the two OMNG, STYH be divided in the ſame manner; as the baſe OMN is to the baſe STY, ſo ſhall the two priſms in the pyramid OMNG be to the two priſms in the pyramid STYH. but the baſe OMN is to the baſe STY, as the baſe ABC to the baſe DEF; therefor as the baſe ABC to



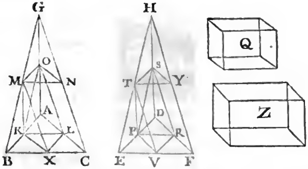
f. 7. 5.

to the baſe DEF, ſo are the two priſms in the pyramid ABCG to the two priſms in the pyramid DEFH; and ſo are the two priſms in the pyramid OMNG to the two priſms in the pyramid STYH; and ſo are all four to all four. and the ſame thing can be ſhewn of the priſms made by dividing the pyramids AKLO and DPRS, and of all made by the ſame number of diviſions. Q. E. D.

PROP. V. THEOR.

PYRAMIDS of the ſame altitude which have triangular baſes, are to one another as their baſes.

Let the pyramids whoſe baſes are the triangles ABC, DEF, and their vertices the points G, H, be of the ſame altitude. as the baſe ABC to the baſe DEF, ſo ſhall the pyramid ABCG be to the pyramid DEFH.



For, if it be not ſo, the baſe ABC ſhall be to the baſe DEF, as the pyramid ABCG to a ſolid either leſs than the pyramid DEFH, or greater than it*. Firſt, let it be to a ſolid leſs than it, viz. to the

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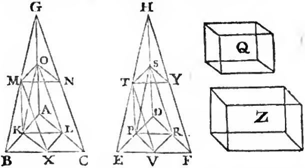
* This can be explained the ſame way as at the note * in Propoſition 2. in the like caſe.

ſolid Q. and divide the pyramid DEFH into two equal pyramids, ſimilar to the whole, and into two equal priſms. theſe two priſms ſhall
a. 3. 12. be greater [a] than the half of the whole pyramid. and, again, let the pyramids made by this diviſion be in like manner divided, and ſo on, until the pyramids which remain undivided in the pyramid DEFH be, all of them together, leſs than the exceſs of the pyramid DEFH above the ſolid Q. let theſe, for example, be the pyramids DPRS, STYH. therefor the priſms, which make the reſt of the pyramid DEFH, are greater than the ſolid Q. divide likewiſe the pyramid ABCG in the



ſame manner, and into as many parts, as the pyramid DEFH. there-
b. 4. 12. for as the baſe ABC to the baſe DEF, ſo [b] are the priſms in the pyramid ABCG to the priſms in the pyramid DEFH. but as the baſe ABC to the baſe DEF, ſo, by Hypotheſis, is the pyramid ABCG to the ſolid Q; and therefor, as the pyramid ABCG to the ſolid Q, ſo are the priſms in the pyramid ABCG to the priſms in the pyramid DEFH. but the pyramid ABCG is greater than the priſms contained
c. 14. 5. in it; wherefor [c] alſo the ſolid Q is greater than the priſms in the pyramid DEFH. but it is alſo leſs, which is impoſſible. therefor the baſe ABC is not to the baſe DEF, as the pyramid ABCG to any ſolid

lid which is leſs than the pyramid DEFH. in the ſame manner it ſhall be demonſtrated, that the baſe DEF is not to the baſe ABC, as the pyramid DEFH to any ſolid which is leſs than the pyramid ABCG. Nor ſhall the baſe ABC be to the baſe DEF, as the pyramid ABCG to any ſolid which is greater than the pyramid DEFH. for, if it be poſſible, let it be to a greater, viz. the ſolid Z. and becauſe the baſe ABC is to the baſe DEF, as the pyramid ABCG to the ſolid Z; by inverſion, as the baſe DEF to the baſe ABC, ſo is the ſolid Z to the pyramid ABCG. but as the ſolid Z is to the pyramid ABCG, ſo ſhall the pyramid DEFH be to ſome ſolid †, which muſt be leſſer [d] than the pyramid ABCG, becauſe the ſolid Z is greater than the pyramid DEFH. and therefor, as the baſe DEF to the baſe ABC, ſo is the pyramid DEFH to a ſolid leſs than the pyramid ABCG; the contrary to which has been proved. therefor the baſe ABC is not to the baſe DEF, as the pyramid ABCG to any ſolid which is greater than the pyramid DEFH. and it has been proved that neither is the baſe ABC to the baſe DEF, as the pyramid ABCG to any ſolid which is leſs than the pyramid DEFH. Therefor as the baſe ABC is to the baſe DEF, ſo is the pyramid ABCG to the pyramid DEFH. Wherefor pyramids &c. Q. E. D.

d. 14. 5.

PROP. VI. THEOR.

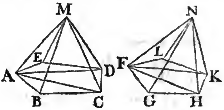
PYRAMIDS of the ſame altitude which have polygons for their baſes, are to one another as their baſes.

Let the pyramids which have the polygons ABCDE, FGHKL for their baſes, and their vertices in the points M, N, be of the ſame altitude

† This can be explained the ſame way as the like at the mark † in Prop. 2.

titude. as the baſe ABCDE to the baſe FGHKL, ſo ſhall the pyramid ABCDEM be to the pyramid FGHKLN.

Divide the baſe ABCDE into the triangles ABC, ACD, ADE; and the baſe FGHKL into the triangles FGH, FHK, FKL. and upon the baſes ABC, ACD, ADE let there be as many pyramids whoſe common vertex is the point M, and upon the remaining baſes as many pyramids having their common vertex in the point N. therefor, be-
a. 5. 12. cauſe the triangle ABC is to the triangle FGH, as [a] the pyramid ABCM to the pyramid FGHN; and the triangle ACD to the triangle FGH, as the pyramid ACDM to the pyramid FGHN; and alſo the triangle ADE to the triangle FGH, as the pyramid ADEM to the pyramid FGHN; as all the firſt antecedents to their common
b. 2. Cor. 24. 5. conſequent, ſo [b] ſhall all the other antecedents be to their common conſequent; that is, as the baſe ABCDE to the baſe FGH, ſo is the pyramid ABCDEM to the pyramid FGHN. and by the ſame reaſon, as the baſe FGHKL to the baſe FGH, ſo is the pyramid FGHKLN to the pyramid FGHN. and, by inverſion, as the baſe FGH to the baſe FGHKL, ſo is the pyramid FGHN to the pyramid FGHKLN. therefor becauſe as the baſe ABCDE to the baſe FGH, ſo is the pyramid ABCDEM to the pyramid FGHN; and as the baſe FGH to the baſe FGHKL, ſo is the pyramid FGHN to the pyramid FGHKLN;
c. 22. 5. it ſhall be, ex aequali [c], as the baſe ABCDE to the baſe FGHKL, ſo



ſo the pyramid ABCDEM to the pyramid FGHKLN. Therefor pyramids &c. Q. E. D. Book XII.

PROP. VII. THEOR.

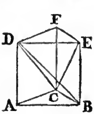
EVERY priſm having a triangular baſe can be divided into three pyramids that have triangular baſes, and are equal to one another.

Let there be a priſm whoſe baſe is the triangle ABC, and DEF the triangle oppoſite to it. the priſm ABCDEF can be divided into three equal pyramids that have triangular baſes.

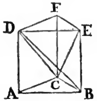
Join BD, EC, CD; and becauſe ABED is a parallelogram whoſe diameter is BD, the triangle ABD ſhall be equal [a] to the triangle EBD; therefor the pyramid whoſe baſe is the triangle ABD, and vertex the point C, is equal [b] to the pyramid whoſe baſe is the triangle EBD, and vertex the point C. but this pyramid is the ſame with the pyramid whoſe baſe is the triangle EBC, and vertex the point D; for they are contained by the ſame planes. therefor the pyramid whoſe baſe is the triangle ABD, and vertex the point C, is equal to the pyramid whoſe baſe is the triangle EBC, and vertex the point D. again, becauſe FCBE is a parallelogram whoſe diameter is CE, the triangle ECF is equal [a] to the triangle ECB; therefor the pyramid whoſe baſe is the triangle ECB, and vertex the point D, is equal to the pyramid whoſe baſe is the triangle ECF, and vertex the point D. but the pyramid whoſe baſe is the triangle ECB, and vertex the point D has been proved equal to the pyramid whoſe baſe is the triangle ABD, and vertex the point C. Therefor the priſm ABCDEF is di-

a. 34. 1.

b. 5. 12.



vided into three equal pyramids which have triangular baſes, viz. into the pyramids ABDC, EBDC, ECFD. and becauſe the pyramid whoſe baſe is the triangle ABD, and vertex the point C, is the ſame with the pyramid whoſe baſe is the triangle ABC, and vertex the point D, for they are contained by the ſame planes; and that the pyramid whoſe baſe is the triangle ABD, and vertex the point C, has been demonſtrated to be a third part of the priſm whoſe baſe is the triangle ABC, and DEF the oppoſite triangle; therefor the pyramid whoſe baſe is the triangle ABC, and vertex the point D, is the third part of the priſm which has the ſame baſe, viz. the triangle ABC, and its oppoſite triangle DEF. Q. E. D.



Cor. 1. From this it is manifeſt that every pyramid is the third part of a priſm which has the ſame baſe, and is of an equal altitude with it; for if the baſe of the priſm be any other figure than a triangle, it may be divided into priſms which have triangular baſes.

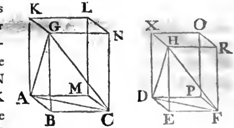
Cor. 2. Priſms whoſe altitudes are equal, are to one another as their baſes; becauſe the pyramids upon the ſame baſes, and of the ſame altitude, are [c] to one another as their baſes. c. 6. 12.

PROP. VIII. THEOR.

SIMILAR pyramids which have triangular baſes, have one to another the triplicate ratio of that which their homologous ſides have.

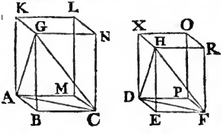
Let the pyramids whoſe baſes are the triangles ABC, DEF, and their vertices the points G, H, be ſimilar and ſimilarly ſituated. the pyramid ABCG ſhall have to the pyramid DEFH, the triplicate ratio of that which the ſide BC has to the homologous ſide EF.

Compleat the parallelograms ABCM, GBCN, ABGK, and the ſolid parallelepiped BGML which is contained by theſe planes and thoſe which are oppoſite to them. and in like manner compleat the ſolid parallelepiped EHPO contained by the three parallelograms DEFP, HEFR, DEHX, and thoſe oppoſite to them. and becauſe the pyramid ABCG is ſimilar to the pyramid DEFH, the angle ABC ſhall be equal [a] to the angle DEF, and the angle GBC to the angle HEF, a. 11. Def. 11. and ABG to DEH. and AB is [b] to BC, as DE to EF; that is, the b. 1. Def. 6. ſides about the equal angles are proportionals; wherefor the parallelogram BM is ſimilar to EP. by the ſame reaſon, the parallelogram BN is ſimilar to ER, and BK to EX. therefor the three parallelograms BM, BN, BK are ſimilar to the three EP, ER, EX. but the three BM, BN, BK are equal and ſimilar [c] to the three which are oppoſite to them, and the three c. 24. 11. EP, ER, EX equal and ſimilar to the three oppoſite to them. wherefor the ſolids BGML, EHPO are contained by the ſame number of ſimilar planes; and their ſolid angles are equal [d]; and therefor the ſolid BGML is d. B. 11. ſimilar [a] to the ſolid EHPO. but ſimilar ſolid parallelepipeds have the triplicate [e] ratio of that which their homologous ſides have. therefor e. 33. 11. the ſolid BGML has to the ſolid EHPO the triplicate ratio of that which the ſide BC has to the homologous ſide EF. but as the ſolid BGML is to the ſolid EHPO, ſo [f] the pyramid ABCG to the pyra- f. 15. 5. mid DEFH; becauſe the pyramids are the ſixth parts of the ſolids, the



the priſm, which is the half[g] of the ſolid parallelepiped, being triple[h] of the pyramid. Wherefor likewiſe the pyramid ABCG has to the pyramid DEFH, the triplicate ratio of that which BC has to the homologous ſide EF. Q. E. D.

Cor. From this it is evident, that ſimilar pyramids which have multangular baſes, are likewiſe to one another in the triplicate ratio of their homologous ſides. for, if they be divided into pyramids that have triangular baſes, becauſe the ſimilar polygons which are their baſes can be divided into the ſame number of ſimilar triangles homologous to the whole polygons; as one of the triangular pyramids in the firſt multangular pyramid is to one of the triangular pyramids in the other, ſo ſhall all the triangular pyramids in the firſt be to all the triangular pyramids in the other; that is, ſo ſhall the firſt multangular pyramid be to the other. but one triangular pyramid is to its ſimilar triangular pyramid, in the triplicate ratio of their homologous ſides; and therefor the firſt multangular pyramid ſhall have to the other, the triplicate ratio of that which one of the ſides of the firſt has to the homologous ſide of the other.



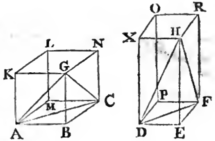
[g] 28. 11.
[h] 7. 12.

PROP. IX.

PROP. IX. THEOR.

THE baſes and altitudes of equal pyramids which have triangular baſes, are reciprocally proportional. and triangular pyramids whoſe baſes and altitudes are reciprocally proportional, are equal to one another.

Let the pyramids whoſe baſes are the triangles ABC, DEF, and which have their vertices in the points G, H be equal to one another. the baſes and altitudes of the pyramids ABCG, DEFH ſhall be reciprocally proportional, viz. the baſe ABC ſhall be to the baſe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG.

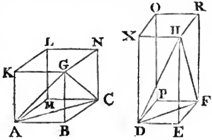


Compleat the parallelograms AC, AG, GC, DF, DH, HF; and the ſolid parallelepipeds BGML, EHPO which are contained by theſe planes and thoſe which are oppoſite to them. and becauſe the pyramid ABCG is equal to the pyramid DEFH, and that the ſolid BGML is ſextuple of the pyramid ABCG, and the ſolid EHPO ſextuple of the pyramid DEFH; the ſolid BGML ſhall be equal [a] to the ſolid EHPO. but the baſes and altitudes of equal ſolid parallelepipeds are reciprocally proportional[b]; therefor as the baſe BM to the baſe EP, ſo is the altitude of the ſolid EHPO to the altitude of the ſolid BGML. but as the baſe BM to the baſe EP, ſo is [c] the triangle ABC to the triangle DEF; therefor as the triangle ABC to the triangle DEF, ſo is the altitude of

a. 1. Ax. 5.
b. 34. 11.
c. 15. 5.

the ſolid EHPO to the altitude of the ſolid BGML. but the altitude of the ſolid EHPO is the ſame with the altitude of the pyramid DEFH; and the altitude of the ſolid BGML is the ſame with the altitude of the pyramid ABCG. therefor, as the baſe ABC to the baſe DEF, ſo is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG. wherefor the baſes and altitudes of the pyramids ABCG, DEFH are reciprocally proportional.

But let the baſes and altitudes of the pyramids ABCG, DEFH be reciprocally proportional, viz. the baſe ABC to the baſe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG. the pyramid ABCG ſhall be equal to the pyramid DEFH.



The ſame conſtruction being made, becauſe as the baſe ABC to the baſe DEF, ſo is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG; and as the baſe ABC to the baſe DEF, ſo is the parallelogram BM to the parallelogram EP; the parallelogram BM ſhall be to EP, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG. but the altitude of the pyramid DEFH is the ſame with the altitude of the ſolid parallelepiped EHPO; and the altitude of the pyramid ABCG is the ſame with the altitude of the ſolid parallelepiped BGML. as, therefor, the baſe BM to the baſe EP, ſo is the altitude of the ſolid parallelepiped EHPO to the altitude of the ſolid parallelepiped BGML. but the ſolid parallelepipeds
b. 34. 11. whoſe baſes and altitudes are reciprocally proportional, are equal [b] to one another. therefor the ſolid parallelepiped BGML is equal to the ſolid

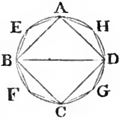
ſolid parallelepiped EHPO. and the pyramid ABCG is the ſixth part of the ſolid BGML, and the pyramid DEFH the ſixth part of the ſolid EHPO. therefor the pyramid ABCG is equal to the pyramid DEFH. Therefor the baſes &c. Q. E. D.

PROP. X. THEOR.

EVERY cone is the third part of a cylinder which has the ſame baſe, and is of an equal altitude with it.

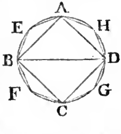
Let a cone have the ſame baſe with a cylinder, viz. the circle ABCD, and the ſame altitude. the cone ſhall be the third part of the cylinder; that is, the cylinder ſhall be triple of the cone.

If the cylinder be not triple of the cone, it ſhall either be greater than the triple, or leſs than it. Firſt, Let it be greater than the triple; and deſcribe the ſquare ABCD in the circle; the ſquare ſhall be greater than the half of the circle ABCD. upon the ſquare ABCD erect a priſm of the ſame altitude with the cylinder; this priſm ſhall be greater than half of the cylinder; becauſe if a ſquare be deſcribed about the circle, and a priſm erected upon the ſquare, of the ſame altitude with the cylinder, the inſcribed ſquare ſhall be half of the circumſcribed; and upon theſe ſquare baſes are erected ſolid parallelepipeds, viz. the priſms, of the ſame altitude; therefor the priſm upon the ſquare ABCD is the half of the priſm upon the ſquare deſcribed about the circle; becauſe they are to one another as their baſes[a]. and the cylinder is leſs than the priſm upon the ſquare deſcribed about the circle ABCD. therefor the priſm upon the ſquare ABCD of the ſame altitude with the cylinder,



[a] 32. 11.

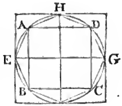
is greater than the half of the cylinder. Biſſect the circumferences AB, BC, CD, DA in the points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA. then, each of the triangles AEB, BFC, CGD, DHA is greater than the half of the ſegment of the circle in which it ſtands, as was ſhewn in Prop. 2. of this Book. Erect priſms upon each of theſe triangles of the ſame altitude with the cylinder; each of theſe priſms ſhall be greater than the half of the ſegment of the cylinder in which it is; becauſe if through the points E, F, G, H parallels be drawn to AB, BC, CD, DA, and parallelograms be compleated upon the ſame AB, BC, CD, DA, and ſolid parallelepipeds be erected upon the parallelograms; the priſms upon the triangles AEB, BFC, CGD, DHA ſhall be the halfs of the ſolid parallelepipeds[b]. and the ſegments of the cylinder which are upon the ſegments of the circle cut off by AB, BC, CD, DA, are leſs than the ſolid parallelepipeds which contain them. therefor the priſms upon the triangles AEB, BFC, CGD, DHA, are greater than the half of the ſegments of the cylinder in which they are. therefor if each of the circumferences be divided into two equal parts, and ſtraight lines be drawn from the points of diviſion to the extremities of the circumferences, and upon the triangles thus made priſms be erected of the ſame altitude with the cylinder, and ſo on, there ſhall at length remain ſome ſegments of the cylinder which together ſhall be leſs[c] than the exceſs of the cylinder above the triple of the cone. let them be theſe upon the ſegments of the circle AE, EB, BF, FC, CG, GD, DH, HA. therefor the reſt of the cylinder, which is the priſm whoſe baſe is the polygon AEBFCGDH, and its altitude the ſame with that of the cylinder, is greater than the triple



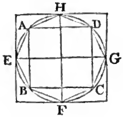
b. 2. Cor. 7. 12.

c. Lemma.

triple of the cone. but this priſm is triple [d] of the pyramid upon the ſame baſe, whoſe vertex is the ſame with the vertex of the cone; d. 1. Cor. 7. 12. therefor the pyramid upon the baſe AEBFCGDH, and which has the ſame vertex with the cone, is greater than the cone, whoſe baſe is the circle ABCD. but it is alſo leſſer, for the pyramid is contained within the cone; which is impoſſible. Nor ſhall the cylinder be leſs than the triple of the cone. let it be leſſer if poſſible. therefor, inverſely, the cone ſhall be greater than the third part of the cylinder. In the circle ABCD deſcribe a ſquare, this ſquare ſhall be greater than the half of the circle. and upon the ſquare ABCD erect a pyramid having the ſame vertex with the cone; this pyramid ſhall be greater than the half of the cone; becauſe, as was before demonſtrated, if a ſquare be deſcribed about the circle, the ſquare ABCD ſhall be the half of it; and if upon theſe ſquares there be erected ſolid parallelepipeds of the ſame altitude with the cone, which are alſo priſms, the priſm upon the ſquare ABCD ſhall be the half of that which is upon the ſquare deſcribed about the circle; for they are to one another as their baſes [a]; as are alſo the third parts of them. therefor the pyramid a. 32. 11. whoſe baſe is the ſquare ABCD is the half of the pyramid upon the ſquare deſcribed about the circle. but this laſt pyramid is greater than the cone which it contains; therefor the pyramid upon the ſquare ABCD whoſe vertex is that of the cone, is greater than the half of the cone. Biſſect the circumferences AB, BC, CD, DA in the points E, F, G, H, and join AE, EB, BF, FC, CG, GD, DH, HA. therefor each of the triangles AEB, BFC, CGD, DHA is greater than half of the ſegment of the circle in which it is. upon each of theſe tri-



angles erect pyramids having the ſame vertex with the cone. each of theſe pyramids ſhall be greater than the half of the ſegment of the cone in which it is, as before was demonſtrated of the priſms and ſegments of the cylinder. and thus dividing each of the circumferences into two equal parts, and joining the points of diviſion and their extremities by ſtraight lines, and upon the triangles erecting pyramids having their vertices the ſame with that of the cone, and ſo on, there ſhall at length remain ſome ſegments of the cone which together ſhall be leſs than the exceſs of the cone above the third part of the cylinder. let theſe be the ſegments upon AE, EB, BF, FC, CG, GD, DH, HA. therefor the reſt of the cone, which is the pyramid whoſe baſe is the polygon AEBFCGDH, and its vertex the ſame with that of the cone, is greater than the third part of the cylinder. but this pyramid is the third part of the priſm upon the ſame baſe AEBFCGDH, and of the ſame altitude with the cylinder. therefor this priſm is greater than the cylinder, whoſe baſe is the circle ABCD. but it is alſo leſſer, for it is contained within the cylinder; which is impoſſible. therefor the cylinder is not leſſer than the triple of the cone. and it has been demonſtrated that neither is it greater than the triple. therefor the cylinder is triple of the cone, or, the cone is the third part of the cylinder. Wherefor every cone &c. Q. E. D.

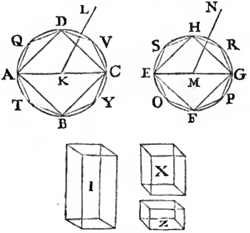


PROP. XI. THEOR.

CONES and cylinders of the ſame altitude, are to one another as their baſes.

Let the cones and cylinders, whoſe baſes are the circles ABCD, EFGH, and their axes KL, MN, and AC, EG the diameters of their baſes, be of the ſame altitude. as the circle ABCD to the circle EFGH, ſo ſhall the cone AL be to the cone EN.

If it be not ſo, the circle ABCD ſhall be to the circle EFGH, as the cone AL to ſome ſolid either leſs than the cone EN, or greater than it. Firſt, let it be to a ſolid leſs than EN, viz. to the ſolid X; and let Z be the ſolid which is equal to the exceſs of the cone EN above the ſolid X; therefor the cone EN is equal to the ſolids X, Z together. in the circle EFGH deſcribe the ſquare EFGH, which ſhall be greater than the half of the circle. upon the ſquare EFGH erect a pyramid of the ſame altitude with the cone; this ſhall be greater than the half of the cone. for if a ſquare be deſcribed about the circle, and a pyramid be erected upon it, of the ſame altitude with the cone*,



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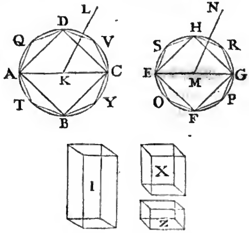
* Rather, having the ſame vertex with the cone, and ſo in ſome places following.

the pyramid inſcribed in the cone ſhall be the half of the pyramid
a. 6. 12. circumſcribed about it, becauſe they are to one another as their baſes [a].
but the cone is leſſer than the circumſcribed pyramid; therefor the
pyramid whoſe baſe is the ſquare EFGH, and its vertex the ſame with
that of the cone, is greater than the half of the cone. divide the circumferences
EF, FG, GH, HE, each into two equal parts in the
points O, P, R, S, and join EO, OF, FP, PG, GR, RH, HS, SE.
therefor each of the triangles EOF, FPG, GRH, HSE is greater
than the half of the ſegment of the circle in which it is. upon each
of theſe triangles erect a pyramid of the ſame altitude with the cone;
each of theſe pyramids is greater than the half of the ſegment of the
cone in which it is. and thus dividing each of theſe circumſerences
into two equal parts, and from the points of diviſion drawing ſtraight
lines to the extremities of the circumferences, and upon each of the
triangles thus made erecting pyramids of the ſame altitude with the
cone, and ſo on, there ſhall at length remain ſome ſegments of the
b. Lemma. cone which ſhall be together leſs [b] than the ſolid Z. let theſe be the
ſegments upon EO, OF, FP, PG, GR, RH, HS, SE. therefor the
remainder of the cone, viz. the pyramid whoſe baſe is the polygon
EOFPGRHS, and its altitude the ſame with that of the cone, is
greater than the ſolid X. In the circle ABCD deſcribe the polygon
ATBYCVDQ ſimilar to the polygon EOFPGRHS, and upon it erect
a pyramid of the ſame altitude with the cone AL. and becauſe
c. 1. 12. as the ſquare of AC is to the ſquare of EG, ſo [c] is the polygon
ATBYCVDQ to the polygon EOFPGRHS; and as the ſquare of
d. 2. 12. AC to the ſquare of EG, ſo is [d] the circle ABCD to the circle EFGH;
e. 11. 5. the circle ABCD ſhall [e] be to the circle EFGH, as the polygon
ATBYCVDQ to the polygon EOFPGRHS. but as the circle ABCD
to the circle EFGH, ſo is the cone AL to the ſolid X; and as the
polygon

polygon ATBYCVDQ to the polygon EOFPGRHS, ſo is [a] the pyramid whoſe baſe is the firſt of theſe polygons, and its vertex L, to the a. 6. 12. pyramid whoſe baſe is the other polygon, and its vertex N. therefor as the cone AL to the ſolid X, ſo is the pyramid whoſe baſe is the polygon ATBYCVDQ, and vertex L to the pyramid whoſe baſe is the polygon EOFPGRHS, and vertex N. but the cone AL is greater than the pyramid contained in it; therefor the ſolid X is greater [f] f. 14. 5.



than the pyramid in the cone EN. but it is leſs, as was ſhewn; which is abſurd. therefor the circle ABCD is not to the circle EFGH, as the cone AL to any ſolid which is leſs than the cone EN. In the ſame manner it ſhall be demonſtrated that the circle EFGH is not to the circle ABCD, as the cone EN to any ſolid leſs than the cone AL. Nor ſhall the circle ABCD be to the circle EFGH, as the cone AL to any ſolid greater than the cone EN. for, if it be poſſible, let it be to the ſolid I which is greater than the cone EN. therefor, by inverſion,

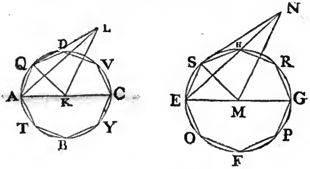
verſion, as the circle EFGH to the circle ABCD, ſo is the ſolid I to
the cone AL. but as the ſolid I to the cone AL, ſo ſhall the cone
f. 14. 5. EN be to ſome ſolid, which muſt be leſs [f] than the cone AL, becauſe
the ſolid I is greater than the cone EN. therefor as the circle EFGH
is to the circle ABCD, ſo is the cone EN to a ſolid which is leſs than
the cone AL, which was ſhewn to be impoſſible. therefor the circle
ABCD is not to the circle EFGH, as the cone AL is to any ſolid
greater than the cone EN. and it has been demonſtrated that neither
is the circle ABCD to the circle EFGH, as the cone AL to any ſolid
leſs than the cone EN. therefor the circle ABCD is to the circle
EFGH, as the cone AL to the cone EN. but as the cone is to the
g. 15. 5. cone, ſo [g] is the cylinder to the cylinder; becauſe the cylinders are
h. 10. 12. triple [h] of the cones, each of each. Therefor as the circle ABCD to
the circle EFGH, ſo are the cylinders upon them of the ſame altitude.
Wherefor cones and cylinders of the ſame altitude, are to one another
as their baſes. Q. E. D.

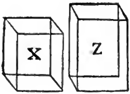
PROP. XII. THEOR.

SIMILAR cones and cylinders have to one another the triplicate ratio of that which the diameters of their baſes have.

Let the cones and cylinders whoſe baſes are the circles ABCD, EFGH, and the diameters of the baſes AC, EG, and KL, MN the axes of the cones or cylinders, be ſimilar. the cone whoſe baſe is the circle ABCD, and vertex the point L, ſhall have to the cone whoſe baſe is the circle EFGH, and vertex N, the triplicate ratio of that which AC has to EG.

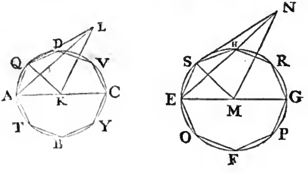
For if the cone ABCDL has not to the cone EFGHN the triplicate ratio of that which AC has to EG, the cone ABCDL ſhall have the triplicate of that ratio to ſome ſolid which is leſs or greater than the cone EFGHN. Firſt, let it have it to a leſs, viz. to the ſolid X. make the ſame conſtruction as in the preceeding Propoſition, and it ſhall be demonſtrated the very ſame way as in that Propoſition, that the pyramid whoſe baſe is the polygon EOFPGRHS, and vertex N

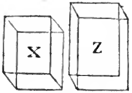




is greater than the ſolid X. Deſcribe alſo in the circle ABCD the polygon ATBYCVDQ ſimilar to the polygon EOFPGRHS, upon which erect a pyramid having the ſame vertex with the cone; and let LAQ be one of the triangles containing the pyramid upon the polygon ATBYCVDQ whoſe vertex is L; and let NES be one of the triangles containing the pyramid upon the polygon EOFPGRHS whoſe vertex is N; and join KQ, MS. becauſe therefor the cone ABCDL

is ſimilar to the cone EFGHN, AC ſhall [a] be to EG, as the axis KL
a. 24. Def. 11. to the axis MN; and as AC to EG, ſo [b] is AK to EM; therefor
b. 15. 5. as AK to EM, ſo is KL to MN; and, alternately, AK to KL, as EM to MN. and the right angles AKL, EMN are equal; therefor, the ſides about theſe equal angles being proportionals, the triangle AKL
c. 6. 6. is ſimilar [c] to the triangle EMN. again, becauſe AK is to KQ, as EM to MS, and that theſe ſides are about equal angles AKQ, EMS, becauſe theſe angles are, each of them, the ſame part of four right angles at the centres K, M; the triangle AKQ ſhall be ſimilar [c] to the triangle EMS. and becauſe it has been ſhewn that as AK to KL, ſo is EM to MN, and that AK is equal to KQ, and EM to MS, as QK to KL, ſo ſhall SM be to MN; and therefor, the ſides about the right angles QKL, SMN being proportionals, the triangle LKQ ſhall





ſhall be ſimilar to the triangle NMS. and becauſe of the ſimilarity of
the triangles AKL, EMN, as LA is to AK, ſo is NE to EM; and by
the ſimilarity of the triangles AKQ, EMS, as KA to AQ, ſo ME
to ES; ex aequali [d] LA ſhall be to AQ, as NE to ES. again, becauſe d. 22. 5.
of the ſimilarity of the triangles LQK, NSM, as LQ to QK, ſo NS to
SM; and from the ſimilarity of the triangles KAQ, MES, as KQ
to QA, ſo MS to SE; ex aequali [d], LQ ſhall be to QA, as NS to
SE. and it was proved that QA is to AL, as SE to EN; therefor,
again, ex aequali, as QL to LA, ſo is SN to NE. wherefor the triangles
LQA, NSE, having the ſides about all their angles proportionals,
ſhall be equiangular [e] and ſimilar to one another. and therefor the e. 5. 6.
pyramid whoſe baſe is the triangle AKQ, and vertex L, is ſimilar to
the pyramid whoſe baſe is the triangle EMS, and vertex N, becauſe
their ſolid angles are equal [f] to one another, and they are contained by f. B. 11.
the ſame number of ſimilar planes. but ſimilar pyramids which have
triangular baſes have to one another the triplicate [g] ratio of that g. 8. 12.
which their homologous ſides have; therefor the pyramid AKQL has
to the pyramid EMSN the triplicate ratio of that which AK has to
EM. In the ſame manner, if ſtraight lines be drawn from the points
D, V, C, Y, B, T to K, and from the points H, R, G, P, F, O to
M, and pyramids be erected upon the triangles having the ſame vertices
with the cones, it ſhall be demonſtrated that each pyramid in the
firſt cone ſhall have to each in the other, taking them in the ſame order,
the triplicate ratio of that which the ſide AK has to the ſide EM;
that is, which AC has to EG. but as one antecedent to its conſequent,
ſo are all the antecedents to all the conſequents [h]; therefor as h. 12. 5.
the pyramid AKQL to the pyramid EMSN, ſo is the whole pyramid
whoſe baſe is the polygon DQATBYCV, and vertex L, to the whole
pyramid whoſe baſe is the polygon HSEOFPGR, and vertex N.

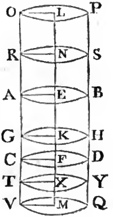
wherefor alſo the firſt of theſe two laſt named pyramids has to the other the triplicate ratio of that which AC has to EG. but, by the Hypotheſis, the cone whoſe baſe is the circle ABCD, and vertex L has to the ſolid X, the triplicate ratio of that which AC has to EG; therefor as the cone whoſe baſe is the circle ABCD, and vertex L, is to the ſolid X, ſo is the pyramid whoſe baſe is the polygon DQATBYCV, and vertex L to the pyramid whoſe baſe is the polygon HSEOFPGR and vertex N. but the ſaid cone is greater than the pyramid contained
i. 14. 5. in it, therefor the ſolid X is greater [i] than the pyramid whoſe baſe is the polygon HSEOFPGR, and vertex N. but it is alſo leſs; which is impoſſible. therefor the cone whoſe baſe is the circle ABCD, and vertex L has not to any ſolid which is leſs than the cone whoſe baſe is the circle EFGH and vertex N, the triplicate ratio of that which AC has to EG. In the ſame manner it ſhall be demonſtrated that neither has the cone EFGHN to any ſolid which is leſs than the cone ABCDL, the triplicate ratio of that which EG has to AC. Nor ſhall the cone ABCDL have to any ſolid which is greater than the cone EFGHN, the triplicate ratio of that which AC has to EG. for, if it be poſſible, let it have it to a greater, viz. to the ſolid Z. therefor, inverſely, the ſolid Z has to the cone ABCDL the triplicate ratio of that which EG has to AC. but as the ſolid Z is to the cone ABCDL, ſo ſhall the cone EFGHN be to ſome ſolid, which muſt be leſs [i] than the cone ABCDL, becauſe the ſolid Z is greater than the cone EFGHN. therefor the cone EFGHN has to a ſolid which is leſs than the cone ABCDL, the triplicate ratio of that which EG has to AC, which was demonſtrated to be impoſſible. therefor the cone ABCDL has not to any ſolid greater than the cone EFGHN, the triplicate ratio of that which AC has to EG; and it was demonſtrated that it could not have that ratio to any ſolid leſs than the cone EFGHN. therefor the

cone ABCDL has to the cone EFGHN, the triplicate ratio of that which AC has to EG. but as the cone is to the cone, ſo [k] the cylinder to the cylinder, for every cone is the third part of the cylinder upon the ſame baſe, and of the ſame altitude. therefor alſo the cylinder has to the cylinder, the triplicate ratio of that which AC has to EG. Wherefor ſimilar cones &c. Q. E. D.

k. 15. 5.

PROP. XIII. THEOR.

IF a cylinder be cut by a plane parallel to its oppoſite planes, or baſes; it ſhall divide the cylinder into two cylinders, one of which ſhall be to the other as the axis of the firſt to the axis of the other.

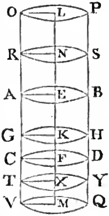


Let the cylinder AD be cut by the plane GH parallel to the oppoſite planes AB, CD, meeting the axis EF in the point K, and let the line GH be the common ſection of the plane GH and the ſurface of the cylinder AD. let AEFC be the parallellogram, in any poſition of it, by the revolution of which about the ſtraight line EF the cylinder AD is deſcribed; and let GK be the common ſection of the plane GH, and the plane AEFC. and becauſe the parallel planes AB, GH are cut by the plane AEKG, AE, KG, their common ſections with it, ſhall be parallel [a]; wherefor AK is a parallelogram, and GK ſhall be equal to EA the ſtraight line from the centre of the circle AB. by the ſame reaſon, each of the ſtraight lines drawn from the point K to the line GH ſhall be proved to be equal

a. 16. 11.

equal to thoſe which are drawn from the centre of the circle AB to its circumference, and ſhall therefor be all of them equal to one another. b. 15. Def. 1. therefor the line GH is the circumference of a circle [b] whoſe centre is the point K. therefor the plane GH divides the cylinder AD into the cylinders AH, GD; for they are the ſame which would be deſcribed by the revolution of the parallelograms AK, GF about the ſtraight lines EK, KF. and the cylinder AH ſhall be to the cylinder HC, as the axis EK to the axis KF.

Produce the axis EF both ways; and take any number of ſtraight lines EN, NL, each equal to EK; and any number FX, XM, each equal to FK; and let planes parallel to AB, CD paſs through the points L, N, X, M. therefor the common ſections of theſe planes with the ſurface of the cylinder produced ſhall be circles whoſe centres are the points L, N X, M; as was proved of the plane GH; and theſe planes ſhall cut off the cylinders PR, RB, DT, TQ. and becauſe their axes LN, NE, EK are c. 11. 12. all equal, the cylinders PR, RB, BG ſhall [c] be to one another as their baſes. but their baſes are equal, and therefor the cylinders PR, RB, BG are equal. and becauſe the axes LN, NE, EK are equal to one another, as alſo the cylinders PR, RB, BG, and that there are as many axes as cylinders; therefor whatever multiple the axis KL is of the axis KE, the ſame multiple is the cylinder PG of the cylinder GB. by the ſame reaſon, what multiple the axis MK is of the axis KF, the ſame multiple is the cylinder QG of the cylinder GD. and if the axis KL be equal to the axis KM, the cylinder PG ſhall be equal to the cylinder GQ; and

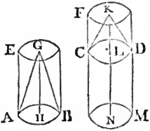


and if the axis KL be greater than the axis KM, the cylinder PG ſhall be greater than the cylinder GQ; and if leſs, leſſer. ſince therefor there are four magnitudes, viz. the axes EK, KF, and the cylinders BG, GD, and that of the axis EK and cylinder BG there has been taken any equimultiples whatever, viz. the axis KL and cylinder PG; and of the axis KF and cylinder GD, any equimultiples whatever, viz. the axis KM and cylinder GQ; and it has been demonſtrated if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if equal, equal; and if leſs, leſſer. therefor [d] the axis EK is to the axis KF, as the cylinder BG to d. 5. Def. 5. the cylinder GD. Wherefor if a cylinder &c. Q. E. D.

PROP. XIV. THEOR.

CONES and cylinders upon equal baſes are to one another as their altitudes.

Let the cylinders EB, FD be upon the equal baſes AB, CD. as the cylinder EB to the cylinder FD, ſo ſhall the axis GH be to the axis KL.

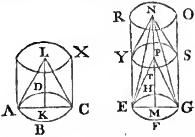


Produce the axis KL to the point N, and make LN equal to the axis GH, and let CM be a cylinder whoſe baſe is CD, and axis LN. and becauſe the cylinders EB, CM have the ſame altitude, they are to one another as their baſes[a]. a. 11. 12. but their baſes are equal, therefor alſo the cylinders EB, CM ſhall be equal. and becauſe the cylinder FM is cut by the plane CD parallel to its oppoſite planes, as the cylinder CM to the cylinder FD, ſo ſhall [b] the axis LN be to b. 13. 12. the axis KL. but the cylinder CM is equal to the cylinder EB, and the axis LN to the axis GH. therefor as the cylinder EB to the cylinder FD, ſo is the axis GH to the axis KL. and as the cylinder c. 15. 5. EB to the cylinder FD, ſo is [c] the cone ABG to the cone CDK, be- d. 10. 12. cauſe the cylinders are triple [d] of the cones. therefor alſo the axis GH is to the axis KL, as the cone ABG to the cone CDK, and the cylinder EB to the cylinder FD. Wherefor cones &c. Q. E. D.

PROP. XV. THEOR.

THE baſes and altitudes of equal cones and cylinders, are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

Let the circles ABCD, EFGH, whoſe diameters are AC, EG, be the baſes, and KL, MN the axes, as alſo the altitudes, of equal cones and cylinders; and let ALC, ENG be the cones, and AX, EO the cylinders. the baſes and altitudes of the cylinders AX, EO ſhall be reciprocally proportional; that is, as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL.



Either the altitude MN is equal to the altitude KL, or the altitudes are not equal. Firſt, let them be equal; and the cylinders AX, EO being alſo equal, and cones and cylinders of the ſame a. 11. 12. altitude being to one another as their baſes[a], therefor the baſe ABCD is equal to the baſe EFGH; and as the baſe ABCD is to the baſe EFGH, ſo ſhall the altitude MN be to the altitude KL. but let the altitudes KL, MN be unequal, and MN the greater of the two, and from MN take MP equal to KL, and, through the point P, cut the cylinder EO by the plane TYS parallel to the oppoſite planes of the circles EFGH, RO; therefor the common ſection of the plane TYS and ſuperficies of the cylinder EO ſhall be a circle, and ES ſhall be a cylinder, whoſe baſe is the circle EFGH, and its altitude MP. and becauſe the cylinder AX is equal to the cylinder EO, as AX is to the cylinder ES, ſo [b] ſhall the cylinder EO be to the ſame ES. but as b. 7. 5. the cylinder AX to the cylinder ES, ſo [a] is the baſe ABCD to the a. 11. 12. baſe EFGH; for the cylinders AX, ES are of the ſame altitude; and as the cylinder EO to the cylinder ES, ſo [c] is the altitude MN to the c. 13. 12. altitude MP, becauſe the cylinder EO is cut by the plane TYS parallel to its oppoſite planes. therefor as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude MP. but MP is equal to the altitude KL; wherefor as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL; that is, the baſes and altitudes of the equal cylinders AX, EO are reciprocally proportional.

But let the baſes and altitudes of the cylinders AX, EO, be reciprocally proportional, viz. the baſe ABCD to the baſe EFGH, as the altitude MN to the altitude KL. the cylinder AX ſhall be equal to the cylinder EO.

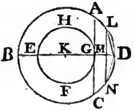
The ſame conſtruction being made, becauſe as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL; and becauſe the altitude KL is equal to the altitude MP; the baſe ABCD ſhall [a] be to the baſe EFGH, as the cylinder AX to the cylinder ES; and as the altitude MN to the altitude MP or KL, ſo is the cylinder EO to the cylinder ES. therefor the cylinder AX is to the cylinder ES, as the cylinder EO is to the ſame ES. whence the cylinder AX is equal to the cylinder EO. and the ſame reaſoning holds in cones. Q. E. D.

PROP. XVI. THEOR.

TO deſcribe in the greater of two circles that have the ſame centre, a polygon of an even number of equal ſides, that ſhall not meet the leſſer circle.

Let ABCD, EFGH be two given circles having the ſame centre K. it is required to inſcribe in the greater circle ABCD a polygon of an even number of equal ſides, that ſhall not meet the leſſer circle.

Through the centre K draw the ſtraight line BD, and from the point G, where it meets the circumference of the leſſer circle, draw GA at right angles to BD, and produce it to C; therefor AC ſhall a. 16. 3. touch [a] the circle EFGH. then if the circumference BAD be biſſected, and the half of it be again biſſected, and ſo on, there ſhall remain at length a circum- b. Lemma. ference leſs [b] than AD. let this be LD; and from the point L draw LM perpendicular to BD, and produce it to N; and join LD, DN. therefor LD is equal to DN, and becauſe LN is parallel to AC, and that AC touches the circle EFGH; LN ſhall not meet the circle EFGH. and much leſs ſhall the ſtraight lines LD, DN meet the circle EFGH. ſo that if ſtraight lines equal to LD be applied in the circle ABCD from the point L around to N, there ſhall be deſcribed in the circle a polygon of an even number of equal ſides that ſhall not meet the leſſer circle. Which was to be done.

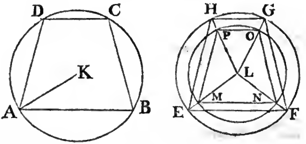


LEMMA II.

LEMMA II.

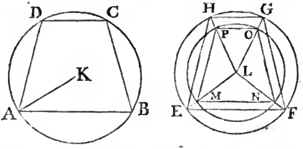
IF two trapeziums ABCD, EFGH be inſcribed in the circles whoſe centres are the points K, L; and if the ſides AB, DC be parallel, as alſo EF, HG; and the other four ſides AD, BC, EH, FG be all equal to one another; but the ſide AB greater than EF, and DC greater than HG. the ſtraight line KA from the centre of the circle in which the greater ſides are, ſhall be greater than the ſtraight line LE from the centre to the circumference of the other circle.

If it be poſſible, let KA be not greater than LE; then KA muſt be either equal to it, or leſſer. Firſt, let KA be equal to LE. therefor becauſe in two equal circles, AD, BC in the one are equal to EH, FG in the other, the circumferences AD, BC ſhall be equal [a] to the circumferences EH, FG; but becauſe the ſtraight lines AB, DC are greater than EF, GH, each than each, the circumferences AB, DC ſhall be greater than EF, HG. therefor the whole circumference ABCD is greater than the whole EFGH; but it is alſo equal to it, a. 28. 3.



which is impoſſible. therefor the ſtraight line KA is not equal to LE.

But let KA be leſs than LE, and make LM equal to KA, and from the centre L, and diſtance LM deſcribe the circle MNOP, meeting the ſtraight lines LE, LF, LG, LH, in M, N, O, P; and join b. 2. 6. MN, NO, OP, PM which ſhall be parallel [b] to, and leſſer than EF, FG, GH, HE. therefor, becauſe EH is greater than MP, AD ſhall be greater than MP; and the circles ABCD, MNOP are equal, therefor the circumference AD is greater than MP; by the ſame reaſon, the circumference BC is greater than NO; and becauſe the ſtraight line AB is greater than EF which is greater than MN, much more ſhall AB be greater than MN. therefor the circumference AB is greater than MN; and by the ſame reaſon, the circumference DC is greater than PO. therefor the whole circumference ABCD is greater than the whole MNOP; but it is likewiſe equal to it, which is impoſſible. therefor KA is not leſſer than LE; nor is it equal to it; the ſtraight line KA muſt therefor be greater than LE. Q. E. D.



Cor. And if there be an Iſoſceles triangle whoſe ſides are equal to AD, BC, but its baſe leſſer than AB the greater of the two ſides AB, DC; the ſtraight line KA ſhall, in the ſame manner, be demonſtrated to be greater than the ſtraight line from the centre to the circumference of the circle deſcribed about the triangle.

PROP. XVII. THEOR.

TO deſcribe in the greater of two ſpheres which have the ſame centre, a ſolid polyhedron, the ſuperficies of which ſhall not meet the leſſer ſphere.

Let there be two ſpheres about the ſame centre A; it is required to deſcribe in the greater a ſolid polyhedron whoſe ſuperficies ſhall not meet the leſſer ſphere.

Let the ſpheres be cut by a plane paſſing thro' the centre; the common ſections of it with the ſpheres ſhall be circles; becauſe the ſphere is deſcribed by the revolution of a ſemicircle about the diameter remaining unmoveable; ſo that in whatever poſition the ſemicircle be conceived, the common ſection of the plane in which it is with the ſuperficies of the ſphere is the circumference of a circle; and this ſhall be a great circle of the ſphere, becauſe the diameter of the ſphere which is likewiſe the diameter of the circle, is greater [a] than any ſtraight line in the circle or ſphere. let then the circle made by the ſection of the plane with the greater ſphere be BCDE, and with the leſſer ſphere be FGH; and draw the two diameters BD, CE at right angles to one another. and in BCDE the greater of the two circles deſcribe [b] a polygon of an even number of equal ſides which ſhall not meet the leſſer circle FGH; and let its ſides, in BE the fourth part of the circle, be BK, KL, LM, ME; join KA and produce it to N; and from A draw AX at right angles to the plane of the circle BCDE meeting the ſuperficies of the ſphere in the point X; and let planes paſs thro' AX and each of the ſtraight lines BD, KN, which,

a. 15. 3.

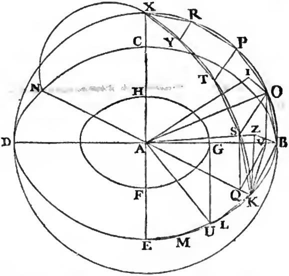
b. 16. 12.

which, from what has been ſaid, ſhall produce great circles on the ſuperficies
of the ſphere, and let BXD, KXN be the ſemicircles thus made
upon the diameters BD, KN. therefor, becauſe XA is at right angles
to the plane of the circle BCDE, every plane which paſſes thro' XA
c. 18. 11. ſhall be at right [c] angles to the plane of the circle BCDE; wherefor
the ſemicircles BXD, KXN ſhall be at right angles to that plane. and
becauſe the ſemicircles BED, BXD, KXN, upon the equal diameters
BD, KN are equal to one another, their fourth parts BE, BX, KX
ſhall be equal to one another. therefor as many ſides of the polygon
as are in the fourth part BE, ſo many ſhall there be in BX, KX equal
to the ſides BK, KL, LM, ME. let theſe polygons be deſcribed, and
their ſides be BO, OP, PR, RX; KS, ST, TY, YX, and join OS,
PT, RY; and from the points O, S draw OV, SQ perpendiculars to
AB, AK. therefor becauſe the plane BOXD is at right angles to the
plane BCDE, and in one of them BOXD, OV is drawn perpendicular
to AB the common ſection of the planes, OV ſhall be perpendicular
d. 4. Def. 11. [d] to the plane BCDE. by the ſame reaſon SQ is perpendicular to
the ſame plane, becauſe the plane KSXN is at right angles to the
plane BCDE. Join VQ, and becauſe in the equal ſemicircles BXD,
KXN the circumferences BO, KS are equal, and OV, SQ are perpendicular
to their diameters, OV ſhall be equal to SQ, and BV equal
to KQ. but the whole BA is equal to the whole KA, therefor the remainder
VA is equal to the remainder QA. as therefor BV is to VA,
e. 2. 6. ſo is KQ to QA, and VQ ſhall be parallel [e] to BK. and becauſe OV,
SQ are each of them at right angles to the plane of the circle BCDE,
f. 6. 11. OV ſhall be parallel [f] to SQ; and it has been proved that it is alſo
g. 33. 1. equal to it; therefor QV, SO are equal and parallel [g]. and becauſe
h. 9. 11. QV is parallel to SO, and alſo to KB; OS ſhall be parallel [h] to
BK; and therefor BO, KS which join them ſhall be in the ſame plane

in which theſe parallels are, and the quadrilateral figure KBOS ſhall be in one plane. and if PB, TK be joined, and perpendiculars be drawn from the points P, T to the ſtraight lines AB, AK, it ſhall be demonſtrated that TP is parallel to KB in the very ſame way that SO was ſhewn to be parallel to the ſame KB; wherefor [h] TP is parallel to SO, and the quadrilateral figure SOPT ſhall be in one plane. by the ſame reaſon the quadrilateral TPRY is in one plane. and the figure YRX is alſo in one plane[i]. therefor, if from the points O, S, P, T, R, Y there be drawn ſtraight lines to the point A, there ſhall be formed a ſolid polyhedron between the circumferences BX, KX compoſed of pyramids whoſe baſes are the quadrilaterals KBOS, SOPT, TPRY, and the triangle YRX, and their common vertex the point A. and if the ſame conſtruction be made upon each of the ſides

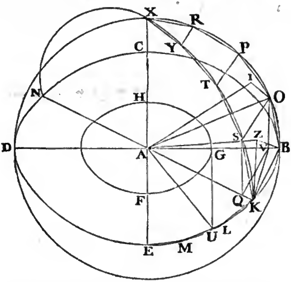
h. 9. 11.

i. 2. 11.



Book XII. KL, LM, ME, as has been done upon BK, and the like be done alſo in the other three quadrants, and in the other hemiſphere; there ſhall be formed a ſolid polyhedron deſcribed in the ſphere, compoſed of pyramids whoſe baſes are the aforeſaid quadrilateral figures, and the triangle YRX, and thoſe formed in the like manner in the reſt of the ſphere, the common vertex of them all being the point A. and the ſuperficies of this ſolid polyhedron ſhall not meet the leſſer ſphere in
k. 11. 11. which is the circle FGH. for from the point A draw [k] AZ perpendicular to the plane of the quadrilateral KBOS meeting it in Z, and join BZ, ZK. and becauſe AZ is perpendicular to the plane KBOS, it ſhall make right angles with every ſtraight line in that plane which meets it; therefor AZ is perpendicular to BZ and ZK. and becauſe AB is equal to AK, and that the ſquares of AZ, ZB, are equal to the ſquare of AB; and the ſquares of AZ, ZK to the ſquare of
* 47. 1. AK*; therefor the ſquares of AZ, ZB are equal to the ſquares of AZ, ZK. take from theſe equals the ſquare of AZ, the remaining ſquare of BZ ſhall be equal to the remaining ſquare of ZK; and therefor the ſtraight line BZ is equal to ZK. in the like manner it ſhall be demonſtrated that the ſtraight lines drawn from the point Z to the points O, S are equal to BZ or ZK. therefor the circle deſcribed from the centre Z, and diſtance ZB ſhall paſs thro' the points K, O, S, and KBOS ſhall be a quadrilateral figure in the circle. and becauſe KB is greater than QV, and QV equal to SO, KB ſhall be greater than SO. but KB is equal to each of the ſtraight lines BO, KS; wherefor each of the circumferences cut off by KB, BO, KS is greater than that cut off by OS; and theſe three circumferences together with a fourth equal to one of them, are greater than the ſame three together with that cut off by OS; that is, than the whole circumference of the circle; therefor the circumference ſubtended by KB is greater than

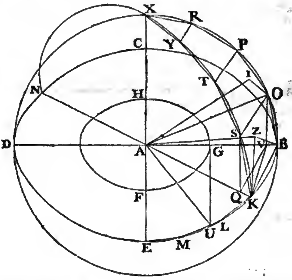
the fourth part of the whole circumference of the circle KBOS, and conſequently the angle BZK at the centre ſhall be greater than a right angle. and becauſe the angle BZK is obtuſe, the ſquare of BK ſhall be greater [l] than the ſquares of BZ, ZK; that is, greater than l. 12. 2. twice the ſquare of BZ. Join KV, and becauſe in the triangles KBV, OBV, KB, BV are equal to OB, BV, and that they contain equal



angles; the angle KVB ſhall be equal [m] to the angle OVB. and OVB m. 4. 1. is a right angle; therefor alſo KVB is a right angle. and becauſe BD is leſſer than twice DV, the rectangle contained by DB, BV is leſſer than twice the rectangle DVB; that is [n], the ſquare of KB is leſs than n. 8. 6. twice the ſquare of KV. but the ſquare of KB is greater than twice the ſquare of BZ; therefor the ſquare of KV is greater than the ſquare of BZ. and becauſe BA is equal to AK, and that the ſquares of BZ, ZA are equal together to the ſquare of BA, and the ſquares of KV, VA to the ſquare of AK; the ſquares of BZ, ZA ſhall be equal to the ſquares of KV, VA; and of theſe the ſquare of KV is greater than the ſquare of BZ, therefor the ſquare of VA ſhall be leſs than the ſquare of ZA, and the ſtraight line AZ greater than VA. much more then ſhall AZ be greater than AG, becauſe in the preceeding Propoſition it was ſhewn that KV falls without the circle FGH. and AZ is perpendicular to the plane KBOS, and is therefor the ſhorteſt of all the ſtraight lines that can be drawn from A the centre of the ſphere to that plane. Therefor the plane KBOS does not meet the leſſer ſphere.

And that the other planes between the quadrants BX, KX fall without the leſſer ſphere, is thus demonſtrated. from the point A draw AI perpendicular to the plane of the quadrilateral SOPT, and join IO; and as was demonſtrated of the plane KBOS and the point Z, in the ſame way it ſhall be ſhewn that the point I is the centre of a circle deſcribed about SOPT, and that OS is greater than PT; and PT was ſhewn to be parallel to OS. therefor becauſe the two trapeziums KBOS, SOPT inſcribed in circles have their ſides BK, OS parallel, as alſo OS, PT; and their other ſides BO, KS, OP, ST all equal to one another, and that BK is greater than OS, and OS greater
a. 2. Lem. 12. than PT, the ſtraight line ZB ſhall be greater [a] than IO. Join AO which ſhall be equal to AB; and becauſe AIO, AZB are right angles, the ſquares of AI, IO ſhall be equal to the ſquare of AO or of AB; that is, to the ſquares of AZ, ZB; and the ſquare of ZB is greater than the ſquare of IO, therefor the ſquare of AZ is leſſer than the ſquare of AI; and the ſtraight line AZ leſſer than the ſtraight line AI. and it was proved that AZ is greater than AG; much more then is AI greater than AG. therefor the plane SOPT falls wholly without

out the leſſer ſphere. in the ſame manner it ſhall be demonſtrated that the plane TPRY falls without the ſame ſphere, as alſo the triangle YRX, viz. by the Cor. of 2d Lemma. and after the ſame way it ſhall be demonſtrated that all the planes which contain the ſolid polyhedron fall without the leſſer ſphere. Therefor in the greater of two ſpheres which have the ſame centre, a ſolid polyhedron is deſcribed



whoſe ſuperficies does not meet the leſſer ſphere. Which was to be done.

But the ſtraight line AZ may be demonſtrated to be greater than AG otherwiſe and in a ſhorter manner, without the help of Prop. 16. as follows. From the point G draw GU at right angles to AG and join AU. if then the circumference BE be biſſected, and its half again biſſected, and ſo on, there ſhall at length be left a circumference leſſer than the circumference which is ſubtended by a ſtraight line equal to GU inſcribed in the circle BCDE. let this be the circumference KB. therefor the ſtraight line KB is leſſer than GU. and becauſe the angle BZK is obtuſe, as was proved in the preceeding, BK ſhall be greater than BZ. but GU is greater than BK; much more then is GU greater than BZ, and the ſquare of GU than the ſquare of BZ. and AU is equal to AB; therefor the ſquare of AU, that is the ſquares of AG, GU are equal to the ſquare of AB, that is to the ſquares of AZ, ZB; but the ſquare of BZ is leſſer than the ſquare of GU; therefor the ſquare of AZ is greater than the ſquare of AG, and the ſtraight line AZ conſequently greater than the ſtraight line AG.

Cor. And if in the leſſer ſphere there be deſcribed a ſolid polyhedron by drawing ſtraight lines betwixt the points in which the ſtraight lines from the centre of the ſphere drawn to all the angles of the ſolid polyhedron in the greater ſphere meet the ſuperficies of the leſſer; in the ſame order in which are joined the points in which the ſame lines from the centre meet the ſuperficies of the greater ſphere; the ſolid polyhedron in the ſphere BCDE ſhall have to this other ſolid polyhedron the triplicate ratio of that which the diameter of the ſphere BCDE has to the diameter of the other ſphere. for if theſe two ſolids be divided into the ſame number of pyramids, and in the ſame order; the pyramids ſhall be ſimilar to one another, each to each. becauſe they have the ſolid angles at their common vertex, the centre of the ſphere, the ſame in each pyramid, and their other ſolid
a. B. 11. angles at the baſes equal to one another, each to each[a], becauſe they are contained by three plane angles equal each to each; and the pyramids are contained by the ſame number of ſimilar planes; and are there-
b. 11. Def. 11. for ſimilar [b] to one another, each to each. but ſimilar pyramids have

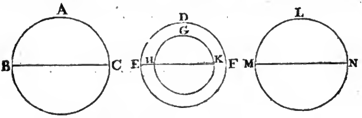
to one another the triplicate [c] ratio of their homologous ſides. therefor the pyramid whoſe baſe is the quadrilateral KBOS, and vertex A, has to the pyramid in the other ſphere of the ſame order, the triplicate ratio of their homologous ſides; that is, of that ratio which AB from the centre of the greater ſphere has to the ſtraight line from the ſame centre to the ſuperficies of the leſſer ſphere. and in like manner each pyramid in the greater ſphere has to each of the ſame order in the leſſer, the triplicate ratio of that which AB has to the ſemidiameter of the leſſer ſphere. and as one antecedent is to its conſequent, ſo are all the antecedents to all the conſequents. Wherefor the whole ſolid polyhedron in the greater ſphere has to the whole ſolid polyhedron in the other, the triplicate ratio of that which AB the ſemidiameter of the firſt has to the ſemidiameter of the other; that is, which the diameter BD of the greater has to the diameter of the other ſphere.

c. Cor. 8. 12.

PROP. XVIII. THEOR.

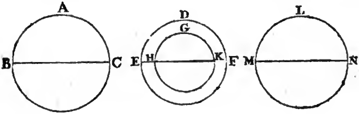
SPHERES have to one another the triplicate ratio of that which their diameters have.

Let ABC, DEF be two ſpheres whoſe diameters are BC, EF.



the ſphere ABC ſhall have to the ſphere DEF the triplicate ratio of that which BC has to EF.

For if it has not, the ſphere ABC ſhall have to a ſphere either leſſer or greater than DEF, the triplicate ratio of that which BC has to EF. Firſt, let it have that ratio to a leſſer, viz. to the ſphere GHK; and let the ſphere DEF have the ſame centre with GHK; a. 17. 12. and in the greater ſphere DEF deſcribe [a] a ſolid polyhedron whoſe ſuperficies ſhall not meet the leſſer ſphere GHK; and in the ſphere ABC deſcribe another ſimilar to that in the ſphere DEF. therefor the ſolid polyhedron in the ſphere ABC ſhall have to the ſolid poly- b. Cor. 17. 11. hedron in the ſphere DEF, the triplicate ratio [b] of that which BC has to EF. but the ſphere ABC has to the ſphere GHK, the triplicate ratio of that which BC has to EF; therefor as the ſphere ABC to the ſphere GHK, ſo is the ſolid polyhedron in the ſphere ABC to the ſolid polyhedron in the ſphere DEF. but the ſphere ABC is greater c. 14. 5. than the ſolid polyhedron in it; therefor [c] alſo the ſphere GHK is greater than the ſolid polyhedron in the ſphere DEF. but it is alſo leſſer, becauſe it is contained within it, which is impoſſible. therefor the ſphere ABC has not to any ſphere leſſer than DEF, the triplicate ratio of that which BC has to EF. In the ſame manner it ſhall be demonſtrated that the ſphere DEF has not to any ſphere leſſer than ABC, the triplicate ratio of that which EF has to BC. Nor ſhall the



the ſphere ABC have to any ſphere greater than DEF, the triplicate ratio of that which BC has to EF. for if it can, let it have that ratio to a greater ſphere LMN. therefor, by inverſion, the ſphere LMN has to the ſphere ABC, the triplicate ratio of that which the diameter EF has to the diameter BC. but as the ſphere LMN to ABC, ſo is the ſphere DEF to ſome ſphere, which muſt be leſſer [c] than the ſphere ABC, becauſe the ſphere LMN is greater than the ſphere DEF. therefor the ſphere DEF has to a ſphere leſſer than ABC the triplicate ratio of that which EF has to BC; which was ſhewn to be impoſſible. therefor the ſphere ABC has not to any ſphere greater than DEF the triplicate ratio of that which BC has to EF. and it was demonſtrated that neither has it that ratio to any ſphere leſſer than DEF. Therefor the ſphere ABC has to the ſphere DEF, the triplicate ratio of that which BC has to EF. Q. E. D.

c. 14. 5.

FINIS.

NOTES

CRITICAL AND GEOMETRICAL;

CONTAINING,

An Account of thoſe Things in which this Edition differs from the Greek Text; and the Reaſons of the Alterations which have been made. As alſo Obſervations on ſome of the Propoſitions.

BY

ROBERT SIMSON, M.D.

Profeſſor of Mathematics in the Univerſity of Glaſgow.

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NOTES &c.

DEFINITION VII. BOOK I.

INSTEAD of this Definition as it is in the Greek copies, a more diſtinct one is given from a property of a plane ſuperficies, which is manifeſtly ſuppoſed in the Elements, viz. that a ſtraight line drawn from any point in a plane to any other in it, is wholly in that plane.

DEF. VIII. B. I.

It ſeems that he who made this Definition deſigned that it ſhould comprehend not only a plane angle contained by two ſtraight lines, but likewiſe the angle which ſome conceive to be made by a ſtraight line and a curve, or by two curve lines, which meet one another in a plane. but tho' the meaning of the words ἐπ' εὐθείας; that is, in a ſtraight line, or in the ſame direction, be plain, when two ſtraight lines are ſaid to be in a ſtraight line, it does not appear what ought to be underſtood by theſe words, when a ſtraight line and a curve, or two curve lines, are ſaid to be in the ſame direction; at leaſt it cannot be explained in this place; which makes it probable that this Definition, and that of the angle of a ſegment, and what is ſaid of the angle of a ſemicircle, and the angles of ſegments, in the 16. and 31. Propoſitions of Book 3. are the additions of ſome leſs ſkilful Editor. on which account, eſpecially ſince they are quite uſeleſs, theſe Definitions are diſtinguiſhed from the reſt by inverted double commas.

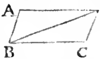
DEF. XVII. BOOK I.

The words "which alſo divides the circle into two equal parts" are added at the end of this Definition in all the copies, but are now left out as not belonging to the Definition, being only a Corollary from it. Proclus demonſtrates it by conceiving one of the parts into which the diameter divides the circle, to be applied to the other, for it is plain they muſt coincide, elſe the ſtraight lines from the centre to the circumference would not be all equal. the ſame thing is eaſily deduced from the 31. Prop. of Book 3. and the 24. of the ſame; from the firſt of which it follows that ſemicircles are ſimilar ſegments of a circle, and from the other, that they are equal to one another.

DEF. XXXIII. B. I.

This Definition has one condition more than is neceſſary; becauſe every quadrilateral figure which has its oppoſite ſides equal to one another, has likewiſe its oppoſite angles equal; and on the contrary.

Let ABCD be a quadrilateral figure, of which the oppoſite ſides AB, CD are equal to one another; as alſo AD and BC. join BD; the two ſides AD, DB are equal to the two CB, BD, and the baſe AB is equal to the baſe CD; therefor by Prop. 8. of Book 1. the angle ADB is equal to the angle CBD; and by Prop. 4. B. 1. the angle BAD ſhall be equal to the angle DCB, and ABD to BDC; and therefor alſo the angle ADC ſhall be equal to the angle ABC.



And if the angle BAD be equal to the oppoſite angle BCD, and the angle ABC to ADC; the oppoſite ſides ſhall be equal. Becauſe by Prop. 32. B. 1. all the angles of the quadrilateral figure ABCD are together equal to four right angles, and the two angles BAD, ADC are together equal to the two angles BCD, ABC. wherefor BAD, ADC are the half of all the four angles; that is, BAD and ADC are equal to two right angles. and therefor AB, CD are parallels by Prop. 28. B. 1. in the ſame manner AD, BC are parallels. therefor ABCD is a parallelogram, and its oppoſite ſides are equal by 34. Prop. B. 1.

PROP. VII. B. I.

There are two caſes of this Propoſition, one of which is not in the Greek text, but is as neceſſary as the other. and that the caſe left out has been formerly in the text appears plainly from this, that the ſecond part of Prop. 5. which is neceſſary to the Demonſtration of this caſe, can be of no uſe at all in the Elements, or any where elſe, but in this Demonſtration; becauſe the ſecond part of Prop. 5. clearly follows from the firſt part, and Prop. 13. B. 1. this part muſt therefor have been added to Prop. 5. upon account of ſome Propoſition betwixt the 5. and 13. but none of theſe ſtand in need of it except the 7. Propoſition, on account of which it has been added. beſides the tranſlation from the Arabic has this caſe explicitely demonſtrated. and Proclus acknowledges that the ſecond part of Prop. 5. was added upon account of Prop. 7. but gives a ridiculous reaſon for it, "that it might "afford an anſwer to objections made againſt the 7." as if the caſe of the 7. which is left out, were, as he expreſly makes it, an objection againſt the Propoſition itſelf. Whoever is curious may read what Proclus ſays of this in his Commentary on the 5. and 7. Propoſitions; for it is not worth while to relate his trifles at full length.

It was thought proper to change the enuntiation of this 7. Prop. ſo as to preſerve the very ſame meaning; the literal tranſlation from the Greek being extremely harſh, and difficult to be underſtood by beginners.

PROP. XI. B. I.

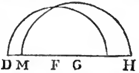
A Corollary is added to this Propoſition, which is neceſſary to Prop. 1. B. 11. and otherways.

PROP. XX. and XXI. B. I.

Proclus in his Commentary relates that the Epicureans derided this Propoſition, as being manifeſt even to aſſes, and needing no Demonſtration; and his anſwer is, that tho' the truth of it be manifeſt to our ſenſes, yet it is ſcience which muſt give the reaſon why two ſides of a triangle are greater than the third. but the right anſwer to this objection againſt this and the 21. and ſome other plain Propoſitions, is, that the number of Axioms ought not to be encreaſed without neceſſity, as it muſt be if theſe Propoſitions be not demonſtrated. Monſ. Clairault in the Preface to his Elements of Geometry publiſhed in French at Paris Ann. 1741. ſays that Euclid has been at the pains to prove that the two ſides of a triangle which is included within another are together leſs than the two ſides of the triangle which includes it; but he has forgot to add this condition, viz. that the triangles muſt be upon the ſame baſe; becauſe without this be added, the ſides of the included triangle may be greater than the ſides of the triangle which includes it, in any ratio which is leſs than that of two to one. as Pappus Alexandrinus has demonſtrated in Prop. 3. B. 3. of his Mathematical Collections.

PROP. XXII. B. I.

Some Authors blame Euclid becauſe he does not demonſtrate that the two circles made uſe of in the conſtruction of this Problem ſhall cut one another. but this is very plain from the determination he has given, viz. that any two of the ſtraight lines DF, FG, GH muſt be greater than the third. for who is ſo dull, tho' only beginning to learn the Elements, as not to perceive that the circle deſcribed from the centre F, at the diſtance FD, muſt meet FH betwixt F and H, becauſe FD is leſſer than FH; and that, for the like reaſon, the circle deſcribed from the centre G, at the diſtance GH or GM muſt meet DG betwixt D and G; and that theſe circles muſt meet one another, becauſe FD and GH are together greater than FG? and this determination is eaſier to be underſtood than that which Mr. Thomas Simpſon derives from it, and puts inſtead of Euclid's, in the 49. page of his Elements of Geometry, that he may ſupply the omiſſion he blames Euclid for; which determination is, that any of the three ſtraight lines muſt be leſſer than the ſum, but greater than the difference of the other two. from this he ſhews the circles muſt meet one another, in one caſe; and ſays that it may be proved after the ſame manner in any other caſe. but the ſtraight line GM which he bids take from GF may be greater than it, as in the figure here annexed, in which caſe his demonſtration muſt be changed into another.

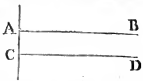


PROP. XXIV. B. I.

To this is added "of the two ſides DE, DF, let DE be that "which is not greater than the other;" that is, take that ſide of the two DE, DF which is not greater than the other, in order to make with it the angle EDG equal to BAC. becauſe without this reſtriction, there might be three different caſes of the Propoſition, as Campanus and others make.

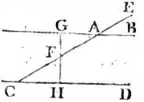
PROP. XXIX. B. I.

The Propoſition which is uſually called the 5. Poſtulate, or 11. Axiom, by ſome the 12. on which this 29. depends, has given a great deal ado to both antient and modern Geometers. it ſeems not to be properly placed among the Axioms, as, indeed, it is not ſelf-evident; nor does it admit of a Demonſtration, in a ſtrict ſenſe; but ſtands in need of ſome explication to make it clear. which, for the ſake of beginners, is given as plainly as we can, as follows.



Firſt, it may eaſily be conceived that two ſtraight lines AB, CD in the ſame plane, which are at right angles to the ſame ſtraight line AC, are likewiſe equidiſtant, that is do neither approach to, nor recede from one another, however far produced; or rather none can conceive the contrary of ſuch lines. for one of them AB cannot be conceived to approach to the other CD, without conceiving AB to incline more to AC upon that ſide of the point A upon which CD is, than it does to it upon the other ſide of A; that is, that AB makes unequal angles with AC. and the ſame thing is to be ſaid of two ſtraight lines which contain any equal angles with a third line, on the ſame ſide and towards the ſame parts of it; for theſe are at right angles to one and the ſame ſtraight line. Let AB, CD contain equal angles EAB, ECD with the ſtraight line EC upon the ſame ſide and towards the ſame parts of it; biſſect AC in F, and draw FG perpendicular to AB, and let it meet CD in H. then becauſe the angle EAB, or FAG [a] is equal to the angle

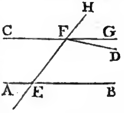
a. 15. 1.



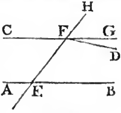
FCH; and the angle AFG equal to CFH[a], and the ſide AF equal a. 15. 1. to the ſide FC in the triangles AFG, CFH; the angle AGF ſhall [b] b. 26. 1. be equal to the angle CHF. but AGF is a right angle, and therefor the angle CHF is a right angle. Wherefor the ſtraight lines BG, DH are at right angles to the ſame ſtraight line GH.

Next, it ſeems to be exceeding manifeſt that two ſtraight lines proceeding from the ſame point, do ſeparate and diverge more and more from one another, ſo as that the neareſt diſtance from the end of one of them to the other may become greater than any given length. if, for inſtance, at the diſtance of ten feet from the point from which they proceed, they be one foot diſtant; when they are each produced another ten feet, their diſtance ſhall be encreaſed by another foot; that is, they ſhall be diſtant by two feet, and ſo more and more. this property flows from the nature or Definition of a ſtraight line which keeps always the ſame direction, and therefor cannot be ſtrictly demonſtrated by the preceeding Propoſitions.

Theſe two things being granted, the Propoſition may be thus demonſtrated. Let two ſtraight lines AB, FD be cut by a third EFH ſo as to make the interior angles upon the ſame ſide of EH, viz. the angles BEF, EFD together leſs than two right angles; AB and FD ſhall meet if produced towards the parts B, D on which theſe angles are.



From the point F draw [a] FG ſo as to a. 23. 1. make the exterior angle HFG equal to BEF the interior and oppoſite on the ſame ſide. therefor EB, FG are parallel[b]; and equidiſtant, as has been explained; and becauſe the angles b. 28. 1. HFG, GFE are together equal [c] to two right angles, the angles BEF, c. 13. 1. EFG ſhall alſo be equal to two right angles. but the angles BEF, EFD are leſſer than two right angles, wherefor EFG is greater than EFD; and FD ſhall be between EB, FG. and the ſtraight lines FG, FD which proceed from the ſame point F, ſhall if produced become further diſtant from one another than FG, EB which keep always the ſame diſtance; and therefor FD ſhall at length be upon the contrary ſide of EB to that on which the point F is; that is, it ſhall meet with and cut EB.



PROP. XXXV. B. I.

The Demonſtration of this Propoſition is changed, becauſe if the method which is uſed in it was followed, there would be three caſes to be ſeparately demonſtrated, as is done in the tranſlation from the Arabic; for in the Elements no caſe of a Propoſition that requires a different Demonſtration ought to be omitted. On this account we have choſen the method which Monſ. Clairault has given, the firſt of any, as far as I know, in his Elements, page 21. and which afterwards Mr. Simpſon gives in his, page 32. but whereas Mr. Simpſon makes uſe of Prop. 26. B. 1. from which the equality of the two triangles does not immediately follow, becauſe to prove that, the 4. of B. 1. muſt likewiſe be made uſe of, as may be ſeen, in the very ſame caſe, in the 34. Prop. B. 1. it was thought better to make uſe only of the 4. of B. 1.

PROP. XLV. B. I.

The ſtraight line KM is proved to be parallel to FL from the 33. Prop. whereas KH is parallel to FG by conſtruction, and KHM, FGL have been demonſtrated to be ſtraight lines. a Corollary is added from Commandine, as being often uſed.

PROP. XIII. B. II.

IN this Propoſition only acute angled triangles are mentioned, whereas it holds true of every triangle. and the Demonſtrations of the caſes omitted are added; Commandine and Clavius have likewiſe given their Demonſtrations of theſe caſes.

PROP. XIV. B. II.

In the Demonſtration of this, ſome Greek Editor has ignorantly inſerted the words, "but if not, one of the two BE, ED ſhall be "the greater; let BE be the greater and produce it to F," as if it was of any conſequence whether the greater or leſſer be produced. therefor inſtead of theſe words, there ought to be read only, "but "if not, produce BE to F."

PROP. I. B. III.

SEVERAL Authors, eſpecially among the modern Mathematicians and Logicians, inveigh too ſeverely againſt indirect, or Apagogic Demonſtrations, and ſometimes ignorantly enough; not being aware that there are ſome things that cannot be demonſtrated any other way. of this the preſent Propoſition is a very clear inſtance, as no direct Demonſtration can be given of it. becauſe, beſides the Definition of a circle, there is no principle or property relating to a circle antecedent to this Problem, from which either a direct or indirect Demonſtration can be deduced. wherefor it is neceſſary that the point found by the conſtruction of the Problem be proved to be the centre of the circle, by the help of this Definition, and ſome of the preceeding Propoſitions. therefor becauſe in the Demonſtration, this Propoſition muſt be brought in, viz. ſtraight lines from the centre of a circle to the circumference are equal, and that the point found by the conſtruction cannot be aſſumed as the centre, for this is the thing to be demonſtrated; it is manifeſt ſome other point muſt be aſſumed as the centre; and if from this aſſumption an abſurdity follows, as Euclid demonſtrates there muſt; then it is not true that the point aſſumed is the centre; and as any point whatever was aſſumed, it follows that no point, except that found by the conſtruction can be the centre. from which the neceſſity of an indirect Demonſtration in this caſe is evident.

PROP. XIII. B. III.

As it is much eaſier to imagine that two circles may touch one another within in more points than one, upon the ſame ſide, than upon oppoſite ſides of them; the figure of that caſe ought not to have been omitted; but the conſtruction in the Greek text would not have ſuted with this figure ſo well, becauſe the centres of the circles muſt have been placed near to the circumferences. on which account another conſtruction and demonſtration is given which is the ſame with the ſecond part of that which Campanus has tranſlated from the Arabic, where without any reaſon the Demonſtration is divided into two parts.

PROP. XV. B. III.

The converſe of the ſecond part of this Propoſition is wanting, tho' in the preceeding, the converſe is added, in a like caſe, both in the Enunciation and Demonſtration; and it is now added in this. beſides in the Demonſtration of the firſt part of this 15th the diameter AD (ſee Commandine's figure) is proved to be greater than the ſtraight line BC by means of another ſtraight line MN; whereas it may be better done without it. on which accounts we have given a different Demonſtration like to that which Euclid gives in the preceeding 14th, and to that which Theodoſius gives, in Prop. 6. B. 1. of his Spherics, in this very affair.

PROP. XVI. B. III.

In this we have not followed the Greek, or the Latin tranſlation literally, but have given what is plainly the meaning of the Propoſition, without mentioning the angle of the ſemicircle, or that which ſome call the cornicular angle which they conceive to be made by the circumference and the ſtraight line which is at right angles to the diameter, at its extremity; which angles have furniſhed matter of great debate between ſome of the modern Geometers, and given occaſion of deducing ſtrange conſequences from them, which are quite avoided by the manner in which we have expreſſed the Propoſition. and in like manner we have given the true meaning of Prop. 31. B. 3. without mentioning the angles of the greater or leſſer ſegments. theſe paſſages Vieta with good reaſon ſuſpects to be adulterated, in the 386. page of his Oper. Math.

PROP. XX. B. III.

The firſt words of the ſecond part of this Demonſtration " κεκλάσθω δὴ πάλιν" are wrong tranſlated by Mr. Briggs and Dr. Gregory " Rurſus inclinetur," for the tranſlation ought to be " Rurſus inflectatur" as Commandine has it. a ſtraight line is ſaid to be inflected either to a ſtraight or curve line, when a ſtraight line is drawn to this line from a point, and from the point in which it meets it, a ſtraight line making an angle with the former is drawn to another point, as is evident from the 90. Prop. of Euclid's Data; for thus the whole line betwixt the firſt and laſt points, is inflected or broken at the point of inflexion where the two ſtraight lines meet. And in the like ſenſe two ſtraight lines are ſaid to be inflected from two points to a third point, when they make an angle at this point; as may be ſeen in the deſcription given by Pappus Alexandrinus of Apollonius Books de Locis planis, in the Preface to his 7. Book. we have made the expreſſion fuller from the 90. Prop. of the Data.

PROP. XXI. B. III.

There are two caſes of this Propoſition, the ſecond of which, viz. when the angles are in a ſegment leſſer than a ſemicircle, is wanting in the Greek. and of this a more ſimple Demonſtration is given than that which is in Commandine, as being derived only from the firſt caſe, without the help of triangles.

PROP. XXIII. and XXIV. B. III.

In Propoſition 24. it is demonſtrated that the ſegment AEB muſt coincide with the ſegment CFD (ſee Commandine's figure) and that it cannot fall otherwiſe, as CGD, ſo as to cut the other circle in a third point G, from this, that if it did, a circle could cut another in more points than two. but this ought to have been proved to be impoſſible in the 23. Prop. as well as that one of the ſegments cannot fall within the other. this part then is left out in the 24. and put in its proper place the 23d Propoſition.

PROP. XXV. B. III.

This Propoſition is divided into three caſes, of which two have the ſame conſtruction and demonſtration; therefor it is now divided only into two caſes.

PROP. XXXIII. B. III.

This alſo in the Greek is divided into three caſes, of which two, viz. one, in which the given angle is acute, and the other in which it is obtuſe, have exactly the ſame conſtruction and demonſtration; on which account the demonſtration of the laſt caſe is left out as quite ſuperfluous, and the addition of ſome unſkilful Editor; beſides the demonſtration of the caſe when the angle given is a right angle, is done a round about way, and is therefor changed to a more ſimple one, as was done by Clavius.

PROP. XXXIV. B. III.

In this and other Propoſitions in which it is required to draw a ſtraight line which ſhall touch a circle, either in any point, or in a given point of the circumference; the thing is done not by the 17. Prop. of this Book, in which the point, from which the tangent is to be drawn, is without the circle; but by drawing from the point in the circumference a ſtraight line at right angles to the diameter paſſing thro' that point, viz. by the 11. Prop. of Book 1. and therefor this Propoſition is cited on the margin in theſe caſes, inſtead of the 17. Prop. of Book 3. which has hitherto been cited in other Editions.

PROP. XXXV. B. III.

As the 25. and 33. Propoſitions are divided into more caſes, ſo this 35. is divided into fewer caſes than are neceſſary. Nor can it be ſuppoſed that Euclid omitted them becauſe they are eaſy; as he has given the caſe which by far is the eaſieſt of them all, viz. that in which both the ſtraight lines paſs thro' the centre. and in the following Propoſition he ſeparately demonſtrates the caſe in which the ſtraight line paſſes thro' the centre, and that in which it does not paſs thro' the centre. ſo that it ſeems Theon, or ſome other, has thought them too long to inſert. but by no means caſes that require different demonſtrations, ſhould be left out in the Elements, as was before taken notice of. theſe caſes are in the tranſlation from the Arabic; and are now put into the Text.

PROP. XXXVII. B. III.

At the end of this the words "in the ſame manner it ſhall be de- "monſtrated, if the centre be in AC" are left out as the addition of ſome ignorant Editor.

DEFINITIONS of BOOK IV.

WHEN a point is in a ſtraight, or any other line, this point is by the Greek Geometers ſaid ἅπτεσθαι, to be upon, or in that line. and when a ſtraight line or circle meets a circle any way, the one is ſaid ἅπτεσθαι to meet the other. but when a ſtraight line or circle meets a circle ſo as not to cut it, it is ſaid ἐφάπτεσθαι, to touch the circle; and theſe two terms are never promiſcuouſly uſed by them. therefor in the 5. Definition of B. 4. the compound ἐφάπτηται muſt be read, inſtead of the ſimple ἅπτηται. and in the 1, 2, 3. and 6. Definitions in Commandine's tranſlation "tangit" muſt be read inſtead of "contingit." and in the 2. and 3. Definitions of Book 3. the ſame change muſt be made. but in the Greek text of Propoſitions 18, 19. B. 3. the compound verb is to be put for the ſimple.

PROP. IV. B. IV.

In this, as alſo in the 8. and 13. Propoſitions of this Book, it is demonſtrated indirectly that the circle touches a ſtraight line; whereas in the 17. 33. and 37. Propoſitions of Book 3. the ſame thing is directly demonſtrated. and this way we have choſen to uſe in the Propoſitions of this Book, as it is ſhorter.

PROP. V. B. IV.

The Demonſtration of this has been ſpoiled by ſome unſkilful hand. for he does not demonſtrate, as is neceſſary, that the two ſtraight lines which biſſect the ſides of the triangle at right angles, ſhall meet one another; and, without any reaſon, he divides the Propoſition into three caſes, whereas one and the ſame conſtruction and demonſtration ſerves for them all, as Campanus has obſerved; which uſeleſs repetitions are now left out. the Greek text alſo in the Corollary is manifeſtly vitiated, where mention is made of a given angle, tho' there is nothing in the Propoſition, nor can be, relating to a given angle.

PROP. XV. and XVI. B. IV.

In the Corollary of the firſt of theſe, the words equilateral and equiangular are wanting in the Greek. and in Prop. 16. inſtead of the circle ABCD ought to be read the circumference ABCD, where mention is made of its containing fifteen equal parts.

DEF. III. B. V.

MANY of the modern Mathematicians reject this Definition as quite wrong. the very learned Dr. Barrow has explained it at large at the end of his third Lecture of the year 1666, in which alſo he anſwers the objections made againſt it as well as the ſubject would allow. and at the end gives his opinion upon the whole, as follows.

" I ſhall only add, that the Author had, perhaps, no other deſign in making this Definition, than (that he might more fully explain and embelliſh his ſubject) to give a general and ſummary idea of ratio to beginners, by premiſing this Metaphyſical Definition, to the more accurate Definitions of ratios that are the ſame to one another, or one of which is greater, or leſſer than the other. I call it a Metaphyſical, for it is not properly a Mathematical Definition, ſince nothing in Mathematics depends on it, or is deduced, nor, as I judge, can be deduced from it. and the Definition of Analogy, which follows, viz. Analogy is the ſimilitude of ratios, is of the ſame kind, and can ſerve for no purpoſe in Mathematics, but only to give beginners ſome general tho' groſs and confuſed notion of Analogy. but the whole of the doctrine of Ratios, and the whole of Mathematics depend upon the accurate Mathematical Definitions which follow this. to theſe we ought principally attend, as the doctrine of Ratios is more perfectly explained by them; this third, and others like it, may quite well be wanted without any loſs to Geometry. as we ſee in the 7. Book of the Elements, where the proportion of numbers to one another is defined, and treated of, yet without giving any Definition of the ratio of numbers. tho' ſuch a Definition was as neceſſary and uſeful to be given in that

" Book, as in this. but indeed there is ſcarce any need of it in either " of them. tho' I think that a thing of ſo general and abſtracted a nature, and thereby the more difficult to be conceived, and explained, cannot be more commodiouſly defined, than as the Author has " done. upon which account I thought fit to explain it at large, and " defend it againſt the captious objections of thoſe who attack it." to this citation from Dr. Barrow I have nothing to add, except that I fully believe the 3. and 8. Definitions are not Euclid's, but added by ſome leſs ſkilful Editor.

DEF. XI. B. V.

It was neceſſary to add the word "continual" before "proportionals" in this Definition; and thus it is cited in the 33. Prop. of Book 11.

After this Definition ought to have followed the Definition of Compound ratio, as this was the proper place for it; Duplicate and Triplicate ratio being ſpecies of Compound ratio. But Theon has made it the 5. Def. of B. 6. where he gives an abſurd and entirely uſeleſs Definition of Compound ratio. for this reaſon we have placed another Definition of it betwixt the 11. and 12. of this Book, which, no doubt, Euclid gave; for he cites it expreſly in Prop. 23. B. 6. and which Clavius, Herigon and Barrow have likewiſe given, but they retain alſo Theon's, which they ought to have left out of the Elements.

DEF. XIII. B. V.

This and the reſt of the Definitions following, contain the explication of ſome terms which are uſed in the 5. and following Books; which, except a few, are eaſily enough underſtood from the Propoſi-

tions

tions of this Book where they are firſt mentioned. they ſeem to have been added by Theon or ſome other. However it be, they are explained ſomething more diſtinctly for the ſake of learners.

PROP. IV. B. V.

In the conſtruction preceeding the demonſtration of this, the words ἃ ἔτυχε, any whatever, are twice wanting in the Greek, as alſo in the Latin tranſlations; and are now added, as being wholly neceſſary.

Ibid. in the demonſtration; in the Greek, and in the Latin tranſlation of Commandine, and in that of Mr. Henry Briggs, which was publiſhed at London in 1620, together with the Greek text of the firſt ſix Books, which in this place is followed by Dr. Gregory in his edition of Euclid, there is this ſentence following, viz. "and of A and C have been taken equimultiples K, L; and of B and D, any equimultiples whatever (ἃ ἔτυχε) M, N;" which is not true. and the the words "any whatever" ought to be left out. and it is ſtrange that neither Mr. Briggs, who did right to leave out theſe words in one place of Prop. 13. of this Book, nor Dr. Gregory who changed them into the word "ſome," or left them out in four places of that ſame Prop. 13. did not alſo leave them out in this place of Prop. 4. and in the ſecond of the two places where they occur in Prop. 17. of this Book, in neither of which they can ſtand conſiſtent with truth. and in none of all theſe places, even in thoſe which they corrected in their Latin tranſlation, have they cancelled the words ἃ ἔτυχε in the Greek text as they ought to have done.

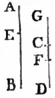
The ſame words ἃ ἔτυχε are found in four places of Prop. 11. of this Book, in the firſt and laſt of which, they are neceſſary, but in the ſecond and third, tho' they are true, they are quite ſuperfluous; as they likewiſe are in the ſecond of the two places in which they are found in the 12. Prop. of this Book.

COR. PROP. IV. B. V.

This Corollary has been unſkilfully annexed to this Propoſition, and has been made inſtead of the legitimate demonſtration which without doubt Theon, or ſome other Editor has taken away, not from this, but from its proper place in this Book. the Author of it deſigned to demonſtrate that if four magnitudes E, G, F, H be proportionals, they ſhall alſo be proportionals inverſely; that is, G ſhall be to E, as H to F; which is true, but the demonſtration of it does not in the leaſt depend upon this 4. Prop. or its demonſtration. for when he ſays "becauſe it is demonſtrated that if K be greater than M, L "ſhall be greater than N" &c. this indeed is ſhewn in the demonſtration of the 4. Prop. but not from this that E, G, F, H are proportionals, for this laſt is the concluſion of the Propoſition. Where for theſe words "becauſe it is demonſtrated" &c. are wholly foreign to his deſign. and he ſhould have proved that if K be greater than M, L ſhall be greater than N, from this, that E, G, F, H are proportionals, and from the 5. Def. of this Book, which he has not; but is done in Propoſition B, which we have given, in its proper place, inſtead of this Corollary. and another Corollary is placed after the 4. Prop. which is often of uſe, and is neceſſary to the Demonſtration of Prop. 18. of this Book.

PROP. V. B. V.

In the conſtruction which precedes the demonſtration of this Propoſition, it is required that EB may be the ſame multiple of CG, that AE is of CF; that is, that EB be divided into as many equal parts, as there are in AE equal to CF. from which it is evident that this conſtruction is not Euclid's. for he does not ſhew the way of dividing ſtraight lines, and far leſs other magnitudes, into any number of equal parts, until the 9. Propoſition of B. 6. and he never requires any thing to be done in the conſtruction, of which he had not before given the method of doing. for this reaſon we have changed the conſtruction to one which without doubt is Euclid's, in which nothing is required but to add a magnitude to itſelf a certain number of times. and this is to be found in the tranſlation from the Arabic, tho' the enunciation of the Propoſition and the demonſtration are there very much ſpoiled. Jacobus Peletarius who was the firſt, as far as I know, who took notice of this error, gives alſo the right conſtruction in his edition of Euclid, after he had given the other he blames, which he ſays he would not leave out, becauſe it was fine, and might ſharpen one's genius to invent others like it; whereas there is not the leaſt difference between the two demonſtrations, except a ſingle word in the conſtruction, which very probably has been owing to an unſkilful Librarian. Clavius likewiſe gives both the ways, but neither he nor Peletarius takes notice of the reaſon why one is preferable to the other.



PROP. VI. B. V.

There are two caſes of this Propoſition, of which only the firſt and ſimpleſt is demonſtrated in the Greek. and it is probable Theon thought it was ſufficient to give this one, ſince he was to make uſe of neither of them in his mutilated Edition of the 5. Book; and he might as well have left out the other, as alſo the 5. Propoſition for the ſame reaſon. the demonſtration of the other caſe is now added, becauſe both of them, as alſo the 5. Propoſition, are neceſſary to the demonſtration of the

18. Prop. of this Book. the tranſlation from the Arabic gives both caſes briefly.

PROP. A. B. V.

This Propoſition is frequently uſed by Geometers, and it is neceſſary in the 25. Prop. of this Book, 31. of the 6. and 34. of the 11. Book. it ſeems to have been taken out of the Elements by Theon, becauſe it appeared evident enough to him, and others who ſubſtitute the confuſed and indiſtinct idea the vulgar have of proportionals, in place of that accurate idea which is to be got from the 5. Def. of this Book. Nor can there be any doubt that Eudoxus or Euclid gave it a place in the Elements, when we ſee the 7. and 9. of the ſame Book demonſtrated, tho' they are quite as eaſy and evident as this. Alphonſus Borellius takes occaſion from this Propoſition to cenſure the 5. Definition of this Book very ſeverely, but moſt unjuſtly. in page 126. of his Euclid reſtored printed at Piſa in 1658. he ſays, "Nor can even this leaſt degree of knowledge be obtained from the foreſaid property," viz. that which is contained in 5. Def. 5. "That if four magnitudes be proportionals, the third ſhall neceſſarily be greater than the fourth, when the firſt is greater than the ſecond; as Clavius acknowledges in the 16. Prop. of the 5. Book of the Elements." But tho' Clavius makes no ſuch acknowledgement expreſly, he has given Borellius a handle to ſay this of him, becauſe when Clavius in the above cited place cenſures Commandine, and that very juſtly, for demonſtrating this Propoſition by help of the 16. of the 5. yet he himſelf gives no demonſtration of it, but thinks it plain from the nature of Proportionals, as he writes in the end of the 14. and 16. Prop. B. 5. of his edition, and is followed by Herigon in Schol. 1. Prop. 14. B. 5. as if there was any nature of Proportionals antecedent to that which is to be derived and underſtood from the Definition of them. and indeed, tho' it is very eaſy to give a right demonſtration of it, no body, as far as I know, has given one, except the learned Dr. Barrow, who, in anſwer to Borellius's objection, demonſtrates it indirectly, but very briefly and clearly from the 5. Definition, in the 322 page of his Lect. Mathem. from which Definition it may alſo be eaſily demonſtrated directly. on which account we have placed it next to the Propoſitions concerning equimultiples.

PROP. B. BOOK. V.

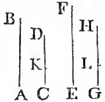
This alſo is eaſily deduced from the 5. Def. B. 5. and therefor is placed next the other, for it was very ignorantly made a Corollary from the 4. Prop. of this Book. See the note on that Corollary.

PROP. C. B. V.

This is frequently made uſe of by Geometers, and is neceſſary to the 5. and 6. Propoſitions of the 10. Book. Clavius in his Notes ſubjoined to the 8. Def. of Book 5. demonſtrates it only in numbers, by help of ſome of the Propoſitions of the 7. Book, in order to demonſtrate the property contained in the 5. Definition of the 5. Book, when applied to numbers, from the property of Proportionals contained in the 20. Def. of the 7. Book. and moſt of the Commentators judge it difficult to prove that four magnitudes which are proportionals according to the 20. Def. of 7. B. are alſo proportionals according to the 5. Def. of 5. Book. but this is eaſily made out, as follows.

Firſt, If A, B, C, D be four magnitudes, ſuch that A is the ſame multiple, or the ſame part of B, which C is of D; A, B, C, D ſhall be proportionals. this is demonſtrated in Propoſition C.

Secondly, If AB contain the ſame parts of CD that EF does of GH; likewiſe in this caſe AB ſhall be to CD, as EF to GH.

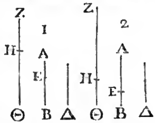


Let CK be a part of CD, and GL the ſame part of GH; and let AB be the ſame multiple of CK, that EF is of GL. therefor by Prop. C of 5. Book, AB is to CK, as EF to GL. and CD, GH are equimultiples of CK, GL the ſecond and fourth; wherefor by Cor. Prop. 4. B. 5. AB is to CD, as EF to GH.

PROP. D. B. V.

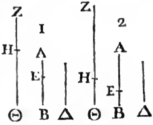
~~This is not unfrequently uſed in the~~ demonſtration of other Propoſitions, and is neceſſary in that of Prop. 9. B. 6. it ſeems Theon has left it out for the reaſon mentioned in the Notes at Prop. A.

PROP. VIII. B. V.



In the demonſtration of this, as it is now in the Greek, there are two caſes, (ſee the demonſtration in Hervagius, or Dr. Gregory's edition) of which the firſt is that in which AE is leſs than EB; and in this, it neceſſarily follows that HΘ the multiple of EB is greater than ZH the ſame multiple of AE, which laſt multiple, by the conſtruction, is greater than Δ; whence alſo HΘ ſhall be greater than Δ. but in the ſecond caſe, viz. that in which EB is leſs than AE, tho' ZH be greater than Δ, yet HΘ may be leſs than the ſame Δ; ſo that there cannot be taken a multiple of Δ which is the firſt that is greater than K, or HΘ, becauſe Δ itſelf is greater than it. upon this account, the

Author of this demonſtration found it neceſſary to change one part of the conſtruction that was made uſe of in the firſt caſe. but he has, without any neceſſity, changed alſo another part of it, viz. when he orders to take N that multiple of Δ which is the firſt that is greater than ZH; for he might have taken that multiple of Δ which is the firſt that is greater than HΘ, or K, as was done in the firſt caſe. he likewiſe brings in this K into the demonſtration of both caſes, without any reaſon, for it ſerves to no purpoſe but to lengthen the demonſtration. There is alſo a third caſe, which is not mentioned in this demonſtration, viz. that in which AE in the firſt, or EB in the ſecond of the two other caſes, is greater than D; and in this any equimultiples, as the doubles, of AE, EB are to be taken, as is done in this Edition, where all the caſes are at once demonſtrated. and from this it is plain that Theon, or ſome other unſkilful Editor has vitiated this Propoſition.



PROP. IX. B. V.

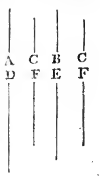
Of this there is given a more explicit demonſtration than that which is now in the Elements.

PROP. X. B. V.

It was neceſſary to give another demonſtration of this Propoſition, becauſe that which is in the Greek, and Latin, or other editions, is not legitimate. for the words *greater*, *the ſame* or *equal*, *leſſer* have a quite different meaning when applied to magnitudes and ratios, as is plain from the 5. and 7. Definitions of B. 5. by the help of theſe let us examine the demonſtration of the 10. Prop. which proceeds thus. " Let A have to C a greater ratio, than B to C. A ſhall " be greater than B. for if it is not greater, it is either equal, or leſs. " but A cannot be equal to B, becauſe then each of them would have " the ſame ratio to C; but they have not. therefor A is not equal to " B." the force of which reaſoning is this, if A had to C, the ſame ratio that B has to C, then if any equimultiples whatever of A and B be taken, and any multiple whatever of C; if the multiple of A be greater than the multiple of C, then, by the 5. Def. of B. 5. the multiple of B ſhall alſo be greater than that of C. but from the Hypotheſis that A has a greater ratio to C, than B has to C, there muſt, by the 7. Def. of B. 5. be certain equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than the ſame multiple of C. and this Propoſition directly contradicts the preceeding; wherefor A is not equal to B. the demonſtration of the 10. Propoſition goes on thus, " but neither is A leſs than B, becauſe then A would have a leſs ratio " to C, than B has to it. but it has not a leſs ratio, therefor A is not " leſs than B" &c. here it is ſaid that " A would have a leſs ratio to " C, than B has to C," or, which is the ſame thing, that B would have a greater ratio to C, than A to C; that is, by 7. Def. B. 5. there muſt be ſome equimultiples of B and A, and ſome multiple of C ſuch, that the multiple of B is greater than the multiple of C, but the multiple of A is not greater than it. and it ought to have been proved that this can never happen if the ratio of A to C, be greater than the ratio of B to C; that is, it ſhould have been proved that in this caſe the multiple of A is ever greater than the multiple of C, whenever the multiple of B is greater than the multiple of C, for when this is demonſtrated it will be evident that B cannot have a greater ratio to C, than A has to C, or, which is the

the ſame thing, that A cannot have a leſſer ratio to C, than B has to C. but this is not at all proved in the 10. Propoſition; but if the 10. were once demonſtrated it would immediately follow from it; but cannot without it be eaſily demonſtrated, as he that tries to do it will find. wherefor the 10. Propoſition is not ſufficiently demonſtrated. and it ſeems that he who has given the demonſtration of the 10. Propoſition as we now have it, inſtead of that which Eudoxus or Euclid had given, has been deceived in applying what is manifeſt when underſtood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and leſſer than another. That thoſe things which are equal to the ſame are equal to one another, is a moſt evident Axiom when underſtood of magnitudes, yet Euclid does not make uſe of it to infer that theſe ratios which are the ſame to the ſame ratio, are the ſame to one another; but explicitly demonſtrates this in Prop. 11. of B. 5. the demonſtration we have given of the 10. Prop. is no doubt the ſame with that of Eudoxus or Euclid, as it is immediately and directly derived from the Definition of a greater ratio, viz. the 7. of the 5.

The above mentioned Propoſition, viz. If A have to C a greater ratio than B to C, and if of A and B there be taken certain equimultiples, and ſome multiple of C, then if the multiple of B be greater than the multiple of C, the multiple of A ſhall alſo be greater than the ſame, is thus demonſtrated.



Let D, E be equimultiples of A, B, and F a multiple of C, ſuch, that E the multiple of B is greater than F; D the multiple of A ſhall alſo be greater than F.

Becauſe A has a greater ratio to C, than B to C, A ſhall be greater than B, by the 10. Prop. B. 5. therefor D the multiple of A ſhall be greater than E the ſame multiple of B. and E is greater than F; therefor much more ſhall D be greater than F.

PROP. XIII. B. V.

In Commandine's, Briggs's and Gregory's Tranſlations, at the beginning of this demonſtration, it is ſaid, “ And the multiple of C is “ greater than the multiple of D; but the multiple of E is not greater “ than the multiple of F,” which words are a literal tranſlation from the Greek. but the ſenſe evidently requires that it be read, “ ſo that “ the multiple of C be greater than the multiple of D; but the multiple of E be not greater than the multiple of F.” and thus this place was reſtored to the true reading in the firſt editions of Commandine's Euclid printed in 8^{vo} at Oxford; but in the later editions, at leaſt in that of 1747, the error of the Greek text was kept in.

There is a Corollary added to Prop. 13. as it is neceſſary to the 20. and 21. Prop. of this Book, and is as uſeful as the Propoſition.

PROP. XIV. B. V.

The two caſes of this which are not in the Greek are added; the demonſtration of them not being exactly the ſame with that of the firſt caſe.

PROP. XVII. B. V.

The order of the words in a clauſe of this is changed to one more natural. as was alſo done in Prop. 11.

PROP. XVIII. B. V.

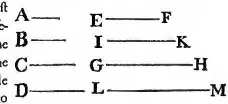
The demonſtration of this is none of Euclid's, nor is it legitimate.

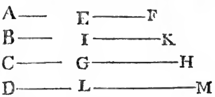
for it depends upon this Hypotheſis, that to any three magnitudes, two of which, at leaſt, are of the ſame kind, there can be a fourth proportional; which if not proved, the Demonſtration now in the text is of no force. but this is aſſumed without any proof, nor can it, as far as I am able to diſcern, be demonſtrated by the Propoſitions preceeding this; ſo far is it from deſerving to be reckoned an Axiom, as Clavius, after other Commentators, would have it, at the end of the Definitions of the 5. Book. Euclid does not demonſtrate it, nor does he ſhew how to find the fourth proportional, before the 12. Prop. of the 6. Book. and he never aſſumes any thing in the demonſtration of a Propoſition, which he had not before demonſtrated; at leaſt, he aſſumes nothing the exiſtence of which is not evidently poſſible; for a certain concluſion can never be deduced by the means of an uncertain Propoſition. upon this account we have given a legitimate Demonſtration of this Propoſition inſtead of that in the Greek and other editions, which very probably Theon, at leaſt ſome other has put in the place of Euclid's, becauſe he thought it too prolix. and as the 17. Prop. of which this 18. is the converſe, is demonſtrated by help of the 1. and 2. Propoſitions of this Book, ſo in the demonſtration now given of the 18th, the 5. Prop. and both caſes of the 6. are neceſſary, and theſe two Propoſitions are the converſes of the 1. and 2. Now the 5. and 6. do not enter into the demonſtration of any Propoſition in this Book as we now have it, nor can they be of uſe in any Propoſition of the Elements, except in this 18. and this is a manifeſt proof that Euclid made uſe of them in his demonſtration of it, and that the demonſtration now given, which is exactly the converſe of that of the 17. as it ought to be, differs nothing from that of Eudoxus or Euclid. for the 5. and 6. have undoubtedly been put into the 5. Book for the ſake of ſome Propoſitions in it, as all the other Propoſitions about equimultiples have been.

Hieronymus Saccherius in his Book named Euclides ab omni naevo vindicatus, printed at Milan Ann. 1733, in 4to, acknowledges this blemiſh in the demonſtration of the 18. and that he may remove it, and render the demonſtration we now have of it legitimate, he endeavours to demonſtrate the following Propoſition, which is in page 115 of his Book, viz.

" Let A, B, C, D be four magnitudes, of which the two firſt are
" of one kind, and alſo the two others either of the ſame kind with
" the two firſt, or of ſome other the ſame kind with one another. I
" ſay the ratio of the third C to the fourth D, is either equal to, or
" greater, or leſſer than the ratio of the firſt A to the ſecond B."
~~And after two Propoſitions premiſed~~ as Lemmas, he proceeds thus.

" Either among all the poſſible equimultiples of the firſt A, and of
" the third C, and, at the ſame time among all the poſſible equimul-
" tiples of the ſecond B, and of the fourth D, there can be found ſome
" one multiple EF of the firſt
" A, and one IK of the ſe-
" cond B, that are equal to one
" another; and alſo (in the
" ſame caſe) ſome one multiple
" GH of the third C equal to
" LM the multiple of the fourth D. or ſuch equality is no where to
" be found. If the firſt caſe happen, [i. e. if ſuch equality is to be
" found,] it is manifeſt from what is before demonſtrated, that A is to
" B, as C to D. but if ſuch ſimultaneous equality be not to be found
" upon both ſides, it will be found either upon one ſide, as upon the
" ſide of A [and B;] or it will be found upon neither ſide; if the
" firſt happen; therefor (from Euclid's Definition of greater and leſ-
" ſer ratio foregoing) A ſhall have to B, a greater or leſſer ratio than
" C to





" C to D; according as GH the multiple of the third C is leſſer, or " greater than LM the multiple of the fourth D. but if " the ſecond caſe happen; therefor upon the one ſide, as upon the ſide of A the firſt and " B the ſecond, it may happen that the multiple EF, [viz. of the firſt] may be leſſer than " IK the multiple of the ſecond, while on the contrary, upon the other " ſide, [viz. of C and D] the multiple GH [of the third C] is greater " than the other multiple LM [of the fourth D.] and then (from " the ſame Definition of Euclid) the ratio of the firſt A to the ſecond " B, ſhall be leſs than the ratio of the third C to the fourth D; or " on the contrary.

" Therefor the Axiom, [i.e. the Propoſition before ſet down,] remains demonſtrated" &c.

Not in the leaſt; but it remains ſtill undemonſtrated; for what he ſays may happen, may in innumerable caſes never happen, and therefor his demonſtration does not hold. for example, if A be the diameter and B the ſide of a ſquare; and C the diameter and D the ſide of another ſquare; there can in no caſe be any multiple of A equal to any of B; nor any one of C equal to one of D, as is well known; and yet it can never happen that when any multiple of A is greater or leſſer than a multiple of B, the multiple of C can, upon the contrary, be leſſer or greater than the multiple of D, viz. taking equimultiples of A and C, and equimultiples of B and D. for A, B, C, D are proportionals, and ſo if the multiple of A be greater &c. than that of B, ſo muſt that of C be greater &c. than that of D. by 5. Def. B. 5.

The ſame objection holds good againſt the Demonſtration which ſome give of the 1. Prop. of the 6. Book, which we have made againſt this of the 18. Propoſition, becauſe it depends upon the ſame inſufficient foundation with the other.

PROP. XIX. B. V.

A Corollary is added to this, which is as frequently uſed as the Propoſition itſelf. the Corollary which is ſubjoined to it in the Greek, plainly ſhews that the 5. Book has been vitiated by Editors who were not Geometers. for the converſion of ratios does nothing depend upon this 19. and the Demonſtration which ſeveral of the Commentators on Euclid give of Converſion, is not legitimate, as Clavius has rightly obſerved, who has given a good Demonſtration of it which we have put in Propoſition E; but he makes it a Corollary from the 19. and begins it with the words, "Hence it eaſily follows," tho' it does not at all follow from it.

PROP. XX, XXI, XXII, XXIII, XXIV. B. V.

The Demonſtrations of the 20. and 21. Propoſitions are ſhorter than thoſe Euclid gives of eaſier Propoſitions, either in the preceeding, or following Books. wherefor it was proper to make them more explicit. and the 22. and 23. Propoſitions, are, as they ought to be, extended to any number of magnitudes. and in like manner may the 24. be, as is taken notice of in a Corollary; and another Corollary is added, as uſeful as the Propoſition. and the words "any whatever" are ſupplied near the end of Prop. 23. which are wanting in the Greek text, and the tranſlations from it.

In a paper wrote by Philippus Naudaeus, and publiſhed, after his death, in the Hiſtory of the Royal Academy of Sciences of Berlin, Ann. 1745. page 50. the 23. Prop. of the 5. Book is cenſured as being obſcurely enuntiated, and, becauſe of this, prolixly demonſtrated. the Enuntiation there given is not Euclid's but Tacquet's, as he acknowledges, which, tho' not ſo well expreſſed, is, upon the matter, the ſame with that which is now in the Elements. Nor is there any thing obſcure in it, tho' the Author of the paper has ſet down the proportionals in a diſadvantageous order, by which it appears to be obſcure. but no doubt Euclid enuntiated this 23. as well as the 22. ſo as to extend it to any number of magnitudes, which taken two and two, are proportionals, and not of ſix only; and to this general caſe the Enuntiation which Naudaeus gives, cannot be well applied.

The Demonſtration which is given of this 23. in that paper, is quite wrong; becauſe if the proportional magnitudes be plane or ſolid figures, there can no rectangle (which he improperly calls a Product) be conceived to be made by any two of them. and if it ſhould be ſaid, that in this caſe ſtraight lines are to be taken which are proportional to the figures, the Demonſtration would this way become much longer than Euclid's. but even tho' his Demonſtration had been right, who does not ſee that it could not be made uſe of in the 5. Book?

PROP. F, G, H. B. V.

Theſe Propoſitions are annexed to the 5. Book, becauſe they are frequently made uſe of by both antient and modern Geometers. and in many caſes Compound ratios cannot be brought into Demonſtrations, without making uſe of them.

Whoever deſires to ſee the doctrine of Ratios delivered in this 5. Book ſolidly defended, and the arguments brought againſt it by And. Tacquet, Alph. Borellius and others, fully refuted, may read Dr. Barrow's Mathematical Lectures, viz. the 7. and 8. of the year 1666.

The 5. Book being thus corrected, I moſt readily agree to what the learned Dr. Barrow ſays*, "That there is nothing in the whole "body of the Elements, of a more ſubtile invention, nothing more ſolidly eſtabliſhed and more accurately handled, than the doctrine of "Proportionals". And there is ſome ground to hope that Geometers will think that this could not have been ſaid with as good reaſon, ſince Theon's time till the preſent.

DEF. II. and V. of B. VI.

THE 2. Definition does not ſeem to be Euclid's but ſome unſkilful Editor's. for there is no mention made by Euclid, nor, as far as I know, by any other Geometer, of reciprocal figures. it is obſcurely expreſſed, which made it proper to render it more diſtinct. it would be better to put the following Definition in place of it, viz.

DEF. II.

Two magnitudes are ſaid to be reciprocally proportional to two others, when one of the firſt is to one of the other magnitudes, as the remaining one of the laſt two is to the remaining one of the firſt.

But the 5. Definition, which ſince Theon's time has been kept in the Elements, to the great detriment of learners, is now juſtly thrown out of them, for the reaſons given in the Notes on the 23. Prop. of this Book.

PROP. I. and II. B. VI.

To the firſt of theſe a Corollary is added which is often uſed. and the Enuntiation of the ſecond is made more general.

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* Page 336.

PROP. III. B. VI.

A ſecond caſe of this, as uſeful as the firſt, is given in Prop A, viz. the caſe in which the exterior angle of a triangle is biſſected by a ſtraight line. the Demonſtration of it is very like to that of the firſt caſe, and upon this account may, probably, have been left out, as alſo the Enuntiation, by ſome unſkilful Editor. at leaſt it is certain, that Pappus makes uſe of this caſe, as an Elementary Propoſition, without a Demonſtration of it, in Prop. 39. of his 7. Book of Mathem. Collections.

PROP. VII. B. VI.

To this a caſe is added which occurs not unfrequently in Demonſtrations.

PROP. VIII. B. VI.

It ſeems plain that ſome Editor has changed the Demonſtration that Euclid gave of this Propoſition. for after he has demonſtrated that the triangles are equiangular to one another, he particularly ſhews that their ſides about the equal angles are proportionals, as if this had not been done in the Demonſtration of the 4. Prop. of this Book. this ſuperfluous part is not found in the Tranſlation from the Arabic, and is now left out.

PROP. IX. B. VI.

This is demonſtrated in a particular caſe, viz. that in which the third part of a ſtraight line is required to be cut off; which is not at all like Euclid's manner. beſides, the Author of the Demonſtration, from four magnitudes being proportionals, concludes that the third of them is the ſame multiple of the fourth, which the firſt is of the ſecond; now this is no where demonſtrated in the 5. Book, as we now have it. but the Editor aſſumes it from the confuſed notion which the Vulgar have of Proportionals. on this account it was neceſſary to give a general and legitimate Demonſtration of this Propoſition.

PROP. XVIII. B. VI.

The Demonſtration of this ſeems to be vitiated. for the Propoſition is demonſtrated only in the caſe of quadrilateral figures, without mentioning how it may be extended to figures of five or more ſides. beſides, from two triangles being equiangular it is inferred that a ſide of the one is to the homologous ſide of the other, as another ſide of the firſt is to the ſide homologous to it of the other, without permutation of the proportionals; which is contrary to Euclid's manner, as is clear from the next Propoſition. and the ſame fault occurs again in the concluſion, where the ſides about the equal angles are not ſhewn to be proportionals; by reaſon of again neglecting permutation. on theſe accounts a Demonſtration is given in Euclid's manner, like to that he makes uſe of in the 20. Prop. of this Book; and it is extended to five ſided figures, by which it may be ſeen how to extend it to figures of any number of ſides.

PROP. XXIII. B. VI.

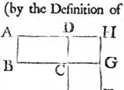
Nothing is uſually reckoned more difficult in the Elements of Geometry by learners, than the doctrine of Compound ratio, which Theon has rendered abſurd and ungeometrical, by ſubſtituting the 5. Definition of the 6. Book, in place of the right Definition which without doubt Eudoxus or Euclid gave, in its proper place, after the Definition of Triplicate ratio &c. in the 5. Book. Theon's Definition is this; a

Ratio is ſaid to be compounded of ratios ὅταν αἱ τῶν λόγων πηλικότητες ἐφ᾽ ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσι τινὰ. which Commandine thus tranſlates, "quando rationum quantitates inter ſe multiplicatae aliquam efficiunt rationem;" that is, when the quantities of the ratios being multiplied by one another make a certain ratio. Dr. Wallis tranſlates the word πηλικότητες, "rationum exponentes," the exponents of the ratios. and Dr. Gregory renders the laſt words of the Definition by "illius facit quantitatem," makes the quantity of that ratio. but in whatever ſenſe the "quantities" or "exponents of the "ratios," and their "multiplication" be taken, the Definition will be ungeometrical and uſeleſs. for there can be no multiplication but by a number; now the quantity or exponent of a ratio (according as Eutocius in his Comment. on Prop. 4. Book 2. of Arch. de Sph. et Cyl. and the moderns explain that term) is the number which multiplied into the conſequent term of a ratio produces the antecedent, or, which is the ſame thing, the number which ariſes by dividing the antecedent by the conſequent; but there are many ratios ſuch, that no number can ariſe from the diviſion of the antecedent by the conſequent; ex. gr. the ratio which the diameter of a ſquare has to the ſide of it; and the ratio which the circumference of a circle has to its diameter, and ſuch like. Beſides, there is not the leaſt mention made of this Definition in the writings of Euclid, Archimedes, Apollonius, or other antients, tho' they frequently make uſe of Compound ratio. and in this 23. Prop. of the 6. Book, where Compound ratio is firſt mentioned, there is not one word which can relate to this Definition, tho' here, if in any place, it was neceſſary to be brought in; but the right Definition is expreſly cited in theſe words, "but the ratio of K to M is compounded "of the ratio of K to L, and of the ratio of L to M." this Definition therefor of Theon is quite uſeleſs and abſurd. for that Theon brought it into the Elements can ſcarce be doubted, as it is to be found in his Commentary upon Ptolomy's Μεγάλη Συντάξις, page 62. where he alſo gives a childiſh explication of it, as agreeing only to ſuch ratios as can be expreſſed by numbers; and from this place the Definition and explication have been exactly copied and prefixed to the Definitions of the 6. Book, as appears from Hervagius's edition. but Zambertus and Commandine in their Latin tranſlations ſubjoin the ſame to theſe Definitions. Neither Campanus, nor, as it ſeems, the Arabic manuſcripts from which he made his Tranſlation, have this Definition. Clavius in his Obſervations upon it, rightly judges that the Definition of Compound ratio might have been made after the ſame manner in which the Definitions of Duplicate and Triplicate ratio are given, viz. "that as in " ſeveral magnitudes that are continual proportionals, Euclid named the " ratio of the firſt to the third, the Duplicate ratio of the firſt to the " ſecond; and the ratio of the firſt to the fourth, the Triplicate ratio of the firſt to the ſecond; that is, the ratio compounded of two " or three intermediate ratios that are equal to one another, and ſo " on; ſo in like manner if there be ſeveral magnitudes of the ſame " kind, following one another, which are not continual proportionals, " the firſt ſhall be ſaid to have to the laſt the ratio compounded of " all the intermediate ratios,——only for this reaſon, that theſe intermediate ratios are interpoſed betwixt the two extremes, viz. the firſt " and laſt magnitudes; even as in the 10. Definition of the 5. Book, " the ratio of the firſt to the third was called the Duplicate ratio, " merely upon account of two ratios being interpoſed betwixt the extremes, that are equal to one another: ſo that there is no difference " betwixt this compounding of ratios, and the duplication or triplication of them which are defined in the 5. Book, but that in the duplication, triplication &c. of ratios, all the interpoſed ratios are equal

" to one another; whereas in the compounding of ratios, it is not " neceſſary that the intermediate ratios ſhould be equal to one another." Alſo Mr. Edmund Scarborough, in his Engliſh tranſlation of the firſt ſix Books, page 238, 266. expreſly affirms that the 5. Definition of the 6. Book, is ſuppoſititious, and that the true Definition of Compound ratio is contained in the 10. Definition of the 5. Book, viz. the Definition of Duplicate ratio, or to be underſtood from it, to wit, in the ſame manner as Clavius has explained it in the preceeding citation. Yet theſe, and the reſt of the Moderns, do notwithſtanding retain this 5. Def. of the 6. B. and illuſtrate and explain it by long Commentaries, when rather they ought to have taken it quite away from the Elements.

For, by comparing Def. 5. B. 6. with Prop. 5. B. 8. it will clearly appear that this Definition has been put into the Elements in place of the right one which has been taken out of them. becauſe in Prop. 5. B. 8. it is demonſtrated that the plane number whoſe ſides are C, D has to the plane number whoſe ſides are E, Z (ſee Hervagius's or Gregory's Edition) the ratio which is compounded of the ratios of their ſides; that is, of the ratios of C to E, and D to Z. and by Def. 5. B. 6. and the explication given of it by all the Commentators, the ratio which is compounded of the ratios of C to E, and D to Z, is the ratio of the product made by the multiplication of the antecedents C, D, to the product of the conſequents E, Z, that is the ratio of the plane number whoſe ſides are C, D to the plane number whoſe ſides are E, Z. wherefor the Propoſition which is the 5. Def. of B. 6. is the very ſame with the 5. Prop. of B. 8. and therefor it ought neceſſarily to be cancelled in one of theſe places; becauſe it is abſurd that the ſame Propoſition ſhould ſtand as a Definition in one place of the Elements, and be demonſtrated in another place of them. Now there is no doubt that Prop. 5. B. 8. ſhould have a place in the Elements, as the ſame thing is demonſtrated in it concerning plane numbers, which is demonſtrated in Prop. 23. B. 6. of equiangular parallelograms; wherefor Def. 5. B. 6. ought not to be in the Elements. and from this it is evident that this Definition is not Euclid's but Theon's, or ſome other unſkilful Geometer's.

But no body, as far as I know, has hitherto ſhewn the true uſe of Compound ratio, or for what purpoſe it has been introduced into Geometry; for every Propoſition in which Compound ratio is made uſe of, may without it be both enuntiated and demonſtrated. Now the uſe of Compound ratio conſiſts wholly in this, that by means of it, circumlocutions may be avoided, and thereby Propoſitions may be more briefly either enuntiated or demonſtrated, or both may be done; for inſtance, if this 23. Propoſition of the 6. Book were to be enuntiated, without mentioning Compound ratio, it might be done as follows; If two parallelograms be equiangular, and if as a ſide of the firſt to a ſide of the ſecond, ſo any aſſumed ſtraight line be made to a ſecond ſtraight line; and as the other ſide of the firſt to the other ſide of the ſecond, ſo the ſecond ſtraight line be made to a third: the firſt parallelogram ſhall be to the ſecond, as the firſt ſtraight line to the third. and the Demonſtration would be exactly the ſame as we now have it. but the antient Geometers, when they obſerved this Enuntiation could be made ſhorter, by giving a name to the ratio which the firſt ſtraight line has to the laſt, by which name the intermediate ratios might likewiſe be ſignified, of the firſt to the ſecond, and of the ſecond to the third, and ſo on if there were more of them, they called this ratio of the firſt to the laſt, the ratio compounded of the ratios of the firſt to the ſecond, and of the ſecond to the third ſtraight line; that is, in the preſent example, of the ratios which are the ſame to the ratios of the ſides. and by this they expreſſed the Propoſition more briefly thus, If there be two equiangular parallelograms, they ſhall have to one another the ratio which is the ſame to that which is compounded of ratios that are the ſame to the ratios of the ſides. which is ſhorter than the preceeding Enuntiation, but has preciſely the ſame meaning. or yet ſhorter thus; equiangular parallelograms have to one another the ratio which is the ſame to that which is compounded of the ratios of their ſides. and theſe two Enuntiations, the firſt eſpecially, agree to the Demonſtration which is now in the Greek. the Propoſition may be more briefly demonſtrated, as Candalla does, thus; Let ABCD, CEFG be two equiangular parallelograms, and compleat the parallelogram CDHG; therefor, becauſe there are three parallelograms AC, CH, CF, the firſt AC ſhall (by the Definition of Compound ratio) have to the third CF, the ratio which is compounded of the ratio of the firſt AC to the ſecond CH, and of the ratio of CH to the third CF; but the parallelogram AC is to the parallelogram CH, as the ſtraight line BC to CG; and the parallelogram CH is to CF, as the ſtraight line CD is to CE; therefor the parallelogram AC has to CF the ratio which is compounded of ratios that are the ſame to the ratios of the ſides. and to this Demonſtration agrees the Enuntiation which is at preſent in the text, viz. equiangular parallelograms have to one another the ratio which is compounded of the ratios of the ſides. for the vulgar reading "which is compounded of their ſides" is abſurd. But in this Edition we have kept the Demonſtration which is in the Greek text, tho' not ſo ſhort as Candalla's; becauſe the way of finding the ratio which is compounded of the ratios of the ſides; that is, of finding the ratio of the parallelograms, is ſhewn in that, but not in Candalla's



dalla's Demonſtration; whereby beginners may learn, in like caſes, how to find the ratio which is compounded of two or more given ratios.

From what has been ſaid it may be obſerved, that in any magnitudes whatever of the ſame kind A, B, C, D &c. the ratio compounded of the ratios of the firſt to the ſecond, of the ſecond to the third, and ſo on to the laſt, is only a name or expreſſion by which the ratio which the firſt A has to the laſt D is ſignified, and by which at the ſame time the ratios of all the magnitudes A to B, B to C, C to D from the firſt to the laſt, to one another, whether they be the ſame, or be not the ſame, are indicated; as in magnitudes which are continual proportionals A, B, C, D &c. the Duplicate ratio of the firſt to the ſecond is only a name, or expreſſion by which the ratio of the firſt A to the third C is ſignified, and by which, at the ſame time, is ſhewn that there are two ratios of the magnitudes from the firſt to the laſt, viz. of the firſt A to the ſecond B, and of the ſecond B to the third or laſt C, which are the ſame to one another; and the Triplicate ratio of the firſt to the ſecond is a name or expreſſion by which the ratio of the firſt A to the fourth D is ſignified, and by which, at the ſame time, is ſhewn that there are three ratios of the magnitudes from the firſt to the laſt, viz. of the firſt A to the ſecond B, and of B to the third C, and of C to the fourth or laſt D, which are all the ſame to one another; and ſo in the caſe of any other Multiplicate ratios. And that this is the right explication of the meaning of theſe ratios is plain from the Definitions of Duplicate and Triplicate ratio in which Euclid makes uſe of the word *λέγεται*, is ſaid to be, or is called; which word, he no doubt made uſe of alſo in the Definition of Compound ratio which Theon, or ſome other, has expunged from the Elements; for the very ſame word is ſtill retained in the wrong Definition of Compound ratio which is now the 5. of the 6. Book. but in the citation of theſe Definitions it is ſometimes retained, as in the Demonſtration of Prop. 19. B. 6. "the firſt is ſaid to have, ἔχειν "λέγεται, to the third the Duplicate ratio" &c. which is wrong tranſlated by Commandine and others "has" inſtead of "is ſaid to have;" and ſometimes it is left out, as in the Demonſtration of Prop. 33. of the 11. Book, in which we find "the firſt has, ἔχει, to the third the "Triplicate ratio;" but without doubt ἔχει, "has," in this place ſignifies the ſame as ἔχειν λέγεται, is ſaid to have. ſo likewiſe in Prop. 23. B. 6. we find this citation "but the ratio of K to M is compounded, σύγκειται, of the ratio of K to L, and the ratio of L to "M," which is a ſhorter way of expreſſing the ſame thing, which, according to the Definition, ought to have been expreſſed by συγκεῖσθαι λέγεται, is ſaid to be compounded.

From theſe Remarks, together with the Propoſitions ſubjoined to the 5. Book, all that is found concerning Compound ratio either in the antient or modern Geometers may be underſtood and explained.

PROP. XXIV. B. VI.

It ſeems that ſome unſkilful Editor has made up this Demonſtration as we now have it, out of two others; one of which may be made from the 2. Prop. and the other from the 4. of this Book. for after he has from the 2. of this Book, and Compoſition and Permutation, demonſtrated that the ſides about the angle common to the two parallelograms are proportionals, he might have immediately concluded that the ſides about the other equal angles were proportionals, viz. from Prop. 34. B. 1. and Prop. 7. B. 5. this he does not, but proceeds to ſhew that the triangles and parallelograms are equiangular, and in a tedious way, by help of Prop. 4. of this Book, and the 22. of B. 5. deduces the ſame concluſion. from which it is plain that this ill compoſed Demonſtration is not Euclid's. theſe ſuperfluous things are now left out, and a more ſimple Demonſtration is given from the 4. Prop. of this Book, the ſame which is in the Tranſlation from the Arabic, by help of the 2. Prop. and Compoſition; but in this the Author neglects Permutation, and does not ſhew the parallelograms to be equiangular, as is proper to do for the ſake of beginners.

PROP. XXV. B. VI.

It is very evident that the Demonſtration which Euclid had given of this Propoſition, has been vitiated by ſome unſkilful hand. for after this Editor had demonſtrated that "as the rectilineal figure ABC "is to the rectilineal KGH, ſo is the parallelogram BE to the paral-"lelogram EF," nothing more ſhould have been added but this, "and "the rectilineal figure ABC is equal to the parallelogram BE, therefor "the rectilineal KGH is equal to the parallelogram EF," viz. from Prop. 14. B. 5. but betwixt theſe two ſentences he has inſerted this, "wherefor, by Permutation, as the rectilineal figure ABC to the pa-"rallelogram BE, ſo is the rectilineal KGH to the parallelogram EF;" by which, it is plain, he thought it was not ſo evident to conclude that the ſecond of four proportionals is equal to the fourth from the equality of the firſt and third, which is a thing demonſtrated in the 14. Prop. of B. 5. as to conclude that the third is equal to the fourth, from the equality of the firſt and ſecond, which is no where demonſtrated in the Elements as we now have them. but tho' this Propoſition, viz. the third of four proportionals is equal to the fourth, if the firſt be equal to the ſecond, had been given in the Elements by Euclid, as very probably it was, yet he would not have made uſe of it in this place, becauſe, as was ſaid, the concluſion could have been immediately deduced without this ſuperfluous ſtep by Permutation. this we have ſhewn at the greater length, both becauſe it affords a certain proof of the vitiation of the Text of Euclid, for the very ſame blunder is found twice in the Greek Text of Prop. 23. B. 11. and twice in Prop. 2. B. 12. and in the 5. 11. 12. and 18. of that Book; in which places of B. 12. except the laſt of them, it is rightly left out in the Oxford Edition of Commandine's Tranſlation: and alſo that Geometers may beware of making uſe of Permutation in the like caſes, for the Moderns not unfrequently commit this miſtake, and among others Commandine himſelf in his Commentary on Prop. 5. B. 3. p. 6. b. of Pappus Alexandrinus, and in other places. the vulgar notion of proportionals has, it ſeems, preoccupied many ſo much, that they do not ſufficiently underſtand the true nature of them.

Beſides, tho' the rectilineal figure ABC, to which another is to be made ſimilar, may be of any kind whatever, yet in the Demonſtration the Greek Text has "triangle" inſtead of "rectilineal figure," which error is corrected in the above named Oxford Edition.

PROP. XXVII. B. VI.

The ſecond Caſe of this has ἀλλῶς, otherwiſe, prefixed to it, as if it was a different Demonſtration, which probably has been done by ſome unſkilful Librarian. Dr. Gregory has rightly left it out. the ſcheme of this ſecond Caſe ought to be marked with the ſame letters of the Alphabet which are in the ſcheme of the firſt, as is now done.

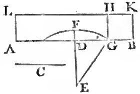
PROP. XXVIII. and XXIX. B. VI.

Theſe two Problems, to the firſt of which the 27. Prop. is neceſſary, are the moſt general and uſeful of all in the Elements, and are moſt frequently made uſe of by the antient Geometers in the ſolution of other Problems; and therefor are very ignorantly left out by Tacquet and Dechales in their Editions of the Elements, who inſipidly ſay they are ſcarce of any uſe. the Caſes of theſe Problems, wherein it is required to apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, either deficient or exceeding by a ſquare; as alſo to apply a rectangle which ſhall be equal to another given, to a given ſtraight line, deficient or exceeding by a ſquare, are very often made uſe of by Geometers. and on this account, it is thought proper, for the ſake of beginners, to give their conſtructions, as follows.

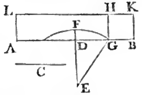
1. To apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, deficient by a ſquare. but the given ſquare muſt not be greater than that upon the half of the given line.

Let AB be the given ſtraight line, and let the ſquare upon the given ſtraight line C be that to which the rectangle to be applied muſt be equal, and this ſquare by the determination, is not greater than that upon half of the ſtraight line AB.

Biſect AB in D, and if the ſquare upon AD be equal to the ſquare upon C, the thing required is done. but if it be not equal to it, AD ſhall be greater than C, according to the determination. draw DE at right angles to AB, and make it equal to C; produce ED to F, ſo that EF be equal to AD or DB, and from the centre E, at the diſtance EF deſcribe a circle meeting AB in G, and upon GB deſcribe the ſquare GBKH, and compleat the rectangle AGHL; alſo join EG. and becauſe AB is biſected in D, the rectangle AG, GB together with the ſquare of DG ſhall be equal [a] to (the ſquare of a. 5. 2.



DB, that is of EF or EG, that is to) the ſquares of ED, DG. take away the ſquare of DG from each of theſe equals, and the remaining rectangle AG, GB ſhall be equal to the ſquare of ED, that is of C. but the rectangle AG, GB is the rectangle AH, becauſe GH is equal to GB. therefor the rectangle AH is equal to the given ſquare upon the ſtraight line C. wherefor the rectangle AH equal to the given ſquare upon C, has been applied to the given ſtraight line AB, deficient by the ſquare GK. Which was to be done.

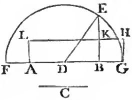


2. To apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, exceeding by a ſquare.

Let AB be the given ſtraight line, and the ſquare upon the given ſtraight line C be that to which the rectangle to be applied muſt be equal.

Biſſect AB in D, and draw BE at right angles to it, ſo that BE be equal to C, and, having joined DE, from the centre D at the diſtance DE deſcribe a circle meeting AB produced in G; upon BG deſcribe the ſquare BGHK, and compleat the rectangle AGHL. and becauſe AB is biſſected in D, and produced to G, the rectangle AG, GB together with the ſquare of DB ſhall be equal [a] to (the ſquare of DG, or DE, that is to) the ſquares of EB, BD. from each of theſe equals take the ſquare of DB, and the remaining rectangle AG, GB ſhall be equal to the ſquare of BE, that is to the ſquare upon C. but the rectangle AG, GB is the rectangle AH, becauſe GH is equal to GB. therefor the rectangle AH is equal to

a. 6. 2.

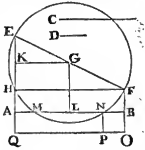


the ſquare upon C. wherefor the rectangle AH equal to the given ſquare upon C, has been applied to the given ſtraight line AB, exceeding by the ſquare GK. Which was to be done.

3. To apply a rectangle to a given ſtraight line which ſhall be equal to a given rectangle, and be deficient by a ſquare. but the given rectangle muſt not be greater than the ſquare upon the half of the given ſtraight line.

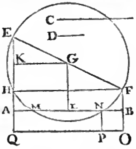
Let AB be the given ſtraight line, and let the given rectangle be that which is contained by the ſtraight lines C, D, which is not greater than the ſquare upon the half of AB. it is required to apply to AB a rectangle equal to the rectangle C, D, deficient by a ſquare.

Draw AE, BF at right angles to AB, upon the ſame ſide of it, and make AE equal to C, and BF to D. join EF and biſſect it in G, and from the centre G, at the diſtance GE deſcribe a circle meeting AE again in H; join HF and draw GK parallel to it, and GL parallel to AE meeting AB in L.



Becauſe the angle EHF in a ſemicircle is equal to the right angle EAB, AB and HF are parallels, and AH and BF are parallels, wherefor AH is equal to BF, and the rectangle EA, AH equal to the rectangle EA, BF, that is to the rectangle C, D. and becauſe EG, GF are equal to one another, and AE, LG, BF parallels, AL and LB ſhall be equal; alſo EK is equal to KH*, * 3. 3.
and the rectangle C, D, from the determination, is not greater than the ſquare of AL the half of AB, wherefor the rectangle EA, AH is not greater than the ſquare of AL, that is of KG. add to each the ſquare of KE, and the ſquare [a] of AK ſhall not be greater than a. 6. 2.

the ſquares of EK, KG, that is than the ſquare of EG; and therefor the ſtraight line AK or GL ſhall not be greater than GE. Now, if GE be equal to GL, the circle EHF ſhall touch AB in L, and the * 36. 3. ſquare of AL ſhall be * equal to the rectangle EA, AH, that is to the given rectangle C, D; and that which was required is done. but if EG, GL be unequal, EG ſhall be the greater, and the circle EHF ſhall cut the ſtraight line AB; let it cut it in the points M, N, and upon NB deſcribe the ſquare NBOP, and compleat the rectangle b. 3. 3. ANPQ. becauſe ML is equal to [b] LN, and it has been proved that AL is equal to LB, AM ſhall be equal to NB. therefor the rectangle AN, NB is equal c. Cor. 36. 3. to the rectangle NA, AM, that is to the rectangle [c] EA, AH or the rectangle C, D. but the rectangle AN, NB is the rectangle AP, becauſe PN is equal to NB. therefor the rectangle AP is equal to the rectangle C, D, and the rectangle AP equal to the given rectangle C, D has been applied to the given ſtraight line AB, deficient by the ſquare BP. Which was to be done.

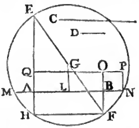


4. To apply a rectangle to a given ſtraight line that ſhall be equal to a given rectangle, exceeding by a ſquare.

Let AB be the given ſtraight line, and the rectangle C, D the given rectangle, it is required to apply a rectangle to AB equal to C, D, exceeding by a ſquare.

Draw AE, BF at right angles to AB, on the contrary ſides of it, and make AE equal to C, and BF equal to D. join EF and biſſect it in G, and from the centre G, at the diſtance GE deſcribe a circle meeting AE again in H; join HF, and draw GL parallel to AE;

let the circle meet AB produced in M, N, and upon BN deſcribe the ſquare NBOP, and compleat the rectangle ANPQ. becauſe the angle EHF in a ſemicircle is equal to the right angle EAB, AB and HF ſhall be parallels, and AH and BF ſhall be equal, alſo the rectangle EA, AH ſhall be equal to the rectangle EA, BF, that is to the rectangle C, D. and becauſe ML is equal to LN, and AL to LB, MA ſhall be equal to BN, and the rectangle AN, NB to MA, AN, that is [a] to the rectangle EA, AH or the rectangle a. 35. 3. C, D. therefor the rectangle AN, NB, that is AP is equal to the rectangle C, D; and to the given ſtraight line AB the rectangle AP has been applied equal to the given rectangle C, D, exceeding by the ſquare BP. Which was to be done.



Willebrordus Snellius was the firſt, as far as I know, who gave theſe conſtructions of the 3. and 4. Problems in his Apollonius Batavus. and afterwards the learned Dr. Halley gave them in the Scholium of the 18. Prop. of the 8. B. of Apollonius's Conics reſtored by him.

The 3. Problem is otherwiſe enuntiated thus, To cut a ſtraight line AB in the point N, ſo as to make the rectangle AN, NB equal to a given ſpace. or, which is the ſame thing, Having given AB the ſum of the ſides of a rectangle, and the magnitude of it being likewiſe given, to find its ſides.

And the 4. Problem is the ſame with this, To find a point N in the given ſtraight line AB produced, ſo as to make the rectangle AN, NB equal to a given ſpace. or, which is the ſame thing, Having given AB the difference of the ſides of a rectangle, and the magnitude of it, to find the ſides.

PROP. XXXI. B. VI.

In the Demonſtration of this the inverſion of proportionals is twice neglected, and is now added, that the concluſion may be legitimately made by help of the 24. Prop. of B. 5. as Clavius had done.

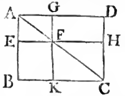
PROP. XXXII. B. VI.

The Enuntiation of the preceeding 26. Prop. is not general enough; becauſe not only two ſimilar parallelograms that have an angle common to both, are about the ſame diameter; but likewiſe two ſimilar parallelograms that have vertically oppoſite angles, have their diameters in the ſame ſtraight line. but there ſeems to have been another, and that a direct Demonſtration of theſe caſes, to which this 32. Propoſition was needful. and the 32. may be otherwiſe and ſomething more briefly demonſtrated as follows.

PROP. XXXII. B. VI.

If two triangles which have two ſides of the one, &c.

Let GAF, HFC be two triangles which have two ſides AG, GF proportional to the two ſides FH, HC, viz. AG to GF, as FH to HC; and let AG be parallel to FH, and GF to HC; AF and FC ſhall be in a ſtraight line.



a. 31. 1. Draw CK parallel [a] to FH, and let it meet
GF produced in K. becauſe AG, KC are each
of them parallel to FH, they ſhall be parallel
b. 30. 1. [b] to one another, and the alternate angles
AGF, FKC ſhall be equal. and AG is to GF, as (FH to HC, that
c. 34. 1. is [c]) CK to KF; wherefor the triangles AGF, CKF are equiangu-
d. 6. 6. lar [d], and the angle AFG ſhall be equal to the angle CFK. but

GFK is a ſtraight line, therefor AF and FC are in a ſtraight line[e]. e. 14. 1.

The 26. Prop. is demonſtrated from the 32. as follows.

If two ſimilar and ſimilarly placed parallelograms have an angle common to both, or vertically oppoſite angles; their diameters ſhall be in the ſame ſtraight line.

Firſt, Let the parallelograms ABCD, AEFG have the angle BAD common to both, and be ſimilar, and ſimilarly placed; ABCD, AEFG ſhall be about the ſame diameter.

Produce EF, GF, to H, K, and join FA, FC. therefor becauſe the parallelograms ABCD, AEFG are ſimilar, DA ſhall be to AB, as GA to AE; wherefor the remainder DG ſhall [a] be to the remainder a. Cor. 19. 5. EB, as GA to AE. but DG is equal to FH, EB to HC, and AE to GF. therefor as FH to HC, ſo is AG to GF; and FH, HC are parallel to AG, GF; and the triangles AGF, FHC are joined at one angle, in the point F; wherefor AF, FC are in the ſame ſtraight line[b]. b. 32. 6.

Next, Let the parallelograms KFHC, GFEA which are ſimilar and ſimilarly placed, have their angles KFH, GFE vertically oppoſite; their diameters AF, FC ſhall be in the ſame ſtraight line.

Becauſe AG, GF are parallel to FH, HC; and that AG is to GF, as FH to HC; AF, FC ſhall be in the ſame ſtraight line[b].

PROP. XXXIII. B. VI.

The words "becauſe they are at the centre," are left out, as the addition of ſome unſkilful hand.

In the Greek, as alſo in the Latin Tranſlation, the words ἃ ἔτυχε, "any whatever," are left out in the Demonſtration of both parts of the Propoſition, and are now added as quite neceſſary. and in the Demonſtration of the ſecond part, where the triangle BGC is proved to be equal to CGK, the illative particle ἄρα in the Greek Text ought to be omitted.

PROP. B, C. B. VI.

Theſe two Propoſitions are added, becauſe they are frequently made uſe of by Geometers.

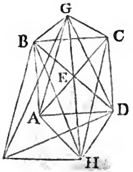
DEF. IX. and XI. B. XI.

THE ſimilitude of plane figures is defined from the equality of their angles, and the proportionality of the ſides about the equal angles; for from the proportionality of the ſides only, or only from the equality of the angles, the ſimilitude of the figures does not follow, except in the caſe when the figures are triangles. the ſimilar poſition of the ſides, which contain the figures, to one another, depending partly upon each of theſe. and, by the ſame reaſon, thoſe are ſimilar ſolid figures which have all their ſolid angles equal, each to each, and are contained by the ſame number of ſimilar plane figures. for there are ſome ſolid figures contained by ſimilar plane figures, of the ſame number, and even of the ſame magnitude, that are neither ſimilar nor equal, as ſhall be demonſtrated after the Notes on the 10. Definition. upon this account it was neceſſary to amend the Definition of ſimilar ſolid figures, and to place the Definition of a ſolid angle before it. and from this and the 10. Definition, it is ſufficiently plain how much the Elements have been ſpoiled by unſkilful Editors.

DEF. X. B. XI.

Since the meaning of the word "equal" is known and eſtabliſhed before it comes to be uſed in this Definition, therefor the Propoſition which is the 10. Definition of this Book, is a Theorem the truth or falſhood of which ought to be demonſtrated, not aſſumed; ſo that Theon, or ſome other Editor, has ignorantly turned a Theorem which ought to be demonſtrated into this 10. Definition. that figures are ſimilar, ought to be proved from the Definition of ſimilar figures; that they are equal ought to be demonſtrated from the Axiom, "Magni-"tudes that wholly coincide, are equal to one another;" or from Prop. A. of Book 5. or the 9. Prop. or the 14. of the ſame Book, from one of which the equality of all kind of figures muſt ultimately be deduced. In the preceeding Books, Euclid has given no Definition of equal figures, and it is certain he did not give this. for what is called the 1. Def. of the 3. Book, is really a Theorem in which theſe circles are ſaid to be equal, that have the ſtraight lines from their centres to the circumferences equal, which is plain from the Definition of a circle, and therefor has by ſome Editor been improperly placed among the Definitions. The equality of figures ought not to be defined, but demonſtrated. therefor tho' it were true that ſolid figures contained by the ſame number of ſimilar and equal plane figures are equal to one another, yet he would juſtly deſerve to be blamed who ſhould make a Definition of this Propoſition which ought to be demonſtrated. But if this Propoſition be not true, muſt it not be confeſſed that Geometers have for theſe thirteen hundred years been miſtaken in this Elementary matter? and this ſhould teach us modeſty, and to acknowledge how little, thro' the weakneſs of our minds, we are able to prevent miſtakes even in the principles of ſciences which are juſtly reckoned amongſt the moſt certain; for that the Propoſition is not univerſally true can by many examples be demonſtrated, the following is ſufficient.

Let ABCD be a ſquare, and draw the diameters AC, BD meeting one another in E; upon one of them BD deſcribe, in the plane in which the ſquare is, an Iſoſceles triangle BFD ſuch, that its vertex be without the ſquare; and from the point E erect the ſtraight line EG at right angles to the plane ABCD, and from any point in it G draw GA, GB, GC, GD, GF. then in the triangles AEG, CEG, becauſe AE, EG are equal to CE, CG, each to each, and that they contain
a. 4. 1. right angles, the baſe AG ſhall [a] be equal to the baſe GC. wherefor in the triangles AGB, CGB, AG, GB are equal to CG, GB, and the
b. 8. 1. baſe AB is equal to the baſe BC; therefor the angle AGB is [b] equal to the angle CGB, and the triangle AGB equal [a] to the triangle CGB. in the ſame manner it ſhall be demonſtrated that the triangle AGD is equal to the triangle CGD. Produce GE to the oppoſite ſide of the plane ABCD, and in it take any point H, and join HA, HB, HC, HD, HF; and, as before, it ſhall be demonſtrated that the triangle AHB is equal to the triangle CHB, and the triangle AHD to the triangle CHD. therefor there are two ſolids, each of which is contained by eight triangles, viz. one contained by four triangles whoſe common vertex is the point G, and their baſes the ſtraight lines BA, AD, BF, FD; and by four other triangles whoſe common vertex is the point H, and their baſes the ſame four ſtraight lines. and the other ſolid is contained by four triangles whoſe common vertex is G, and their baſes the ſtraight lines BC, CD, BF, FD; and by four other triangles, whoſe common vertex is the point H, and baſes the ſame ſtraight lines BC, CD, BF, FD. but the four triangles



triangles AGB, AGD, AHB, AHD have been demonſtrated to be equal to the four triangles CGB, CGD, CHB, CHD, each to each; and the four other triangles BGF, DGF, BHF, DHF are common to both ſolids. therefor theſe two ſolids are contained by the ſame number of ſimilar and equal plane figures. and that they are not equal to one another is manifeſt, becauſe the firſt of them is contained within the other. it is not therefor univerſally true that thoſe ſolids are equal which are contained by the ſame number of ſimilar and equal plane figures.

COR. Hence it is manifeſt that two unequal ſolid angles may be contained by the ſame number of plane angles which are equal to one another, each to each.

For the ſolid angle at G which is contained by the four plane angles AGB, AGD, FGB, FGD, is plainly unequal to the ſolid angle at the ſame point G, which is contained by the four plane angles CGB, CGD, FGB, FGD; becauſe this laſt contains the other. and each of them is contained by four plane angles, which are equal to one another, each to each, or are the ſelf ſame; as has been proved. and, indeed, there may be innumerable ſolid angles all unequal to one another, which are each of them contained by plane angles that are equal to one another, each to each. it is likewiſe manifeſt that the before mentioned ſolids are not at all ſimilar, ſince their ſolid angles are not all equal.

And that there may be innumerable ſolid angles all unequal to one another, which are each of them contained by the ſame plane angles diſpoſed in the ſame order, will be plain from the three following Propoſitions.

PROP. I. PROBLEM.

Three magnitudes A, B, C being given, to find a fourth ſuch, that every three ſhall be greater than the remaining one.

Let D be the fourth, therefor D ſhall be leſſer than A, B, C together. of the three A, B, C let A be that which is not leſſer than either of the two B and C. and firſt, let B and C together be not leſſer than A; therefor B, C, D together ſhall be greater than A. and becauſe A is not leſſer than B; A, C, D together ſhall be greater than B. in the like manner A, B, D together ſhall be greater than C. wherefor in the caſe in which B and C together are not leſſer than A, any magnitude D which is leſſer than A, B, C together will anſwer the Problem.

But if B and C together be leſſer than A, then becauſe it is required that B, C, D together be greater than A, from each of theſe taking away B, C, the remaining one D ſhall be greater than the exceſs of A above B and C. take therefor any magnitude D which is leſs than A, B, C together, but greater than the exceſs of A above B and C. then B, C, D together ſhall be greater than A; and becauſe A is greater than either B or C, much more ſhall A and D, together with either of the two B, C be greater than the other. and, by the conſtruction, A, B, C are together greater than D.

Cor. If beſides, it be required that A and B together ſhall not be leſſer than C and D together; the exceſs of A and B together above C muſt not be leſſer than D, that is D muſt not be greater than that exceſs.

PROP. II. PROBLEM.

Four magnitudes A, B, C, D being given of which A and B together are not leſſer than C and D together, and ſuch that any three of them whatever are greater than the fourth; it is required to find a fifth magnitude E ſuch, that any two of the three A, B, E ſhall be greater than the third, and alſo that any two of the three C, D, E ſhall be greater than the third. Let A be not leſſer than B, and C not leſſer than D.

Firſt, Let the exceſs of C above D be not leſſer than the exceſs of A above B. it is plain that a magnitude E can be taken which is leſſer than the ſum of C and D, but greater than the exceſs of C above D; let it be taken, then ſhall E be greater likewiſe than the exceſs of A above B; wherefor E and B together ſhall be greater than A; and A is not leſſer than B, therefor A and E together are greater than B. and, by the Hypotheſis, A and B together are not leſſer than C and D together, and C and D together are greater than E; therefor likewiſe A and B are greater than E.

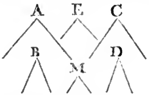
But let the exceſs of A above B be greater than the exceſs of C above D. and, becauſe, by the Hypotheſis, the three B, C, D are together greater than the fourth A; C and D together ſhall be greater than the exceſs of A above B. therefor a magnitude can be taken which is leſſer than C and D together, but greater than the exceſs of A above B. Let this magnitude be E, and becauſe E is greater than the exceſs of A above B, B together with E ſhall be greater than A. and, as in the preceeding caſe, it ſhall be ſhewn that A together with E is greater than B, and that A together with B is greater than E. therefor in each of the caſes it has been ſhewn that any two of the three A, B, E are greater than the third.

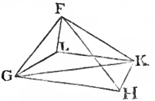
And becauſe in each of the caſes E is greater than the exceſs of C above D, E together with D ſhall be greater than C, and, by the Hypotheſis, C is not leſſer than D, therefor E together with C is greater than D; and, by the conſtruction, C and D together are greater than E. therefor any two of the three, C, D, E are greater than the third.

PROP. III. THEOREM.

There may be innumerable ſolid angles all unequal to one another, each of which is contained by the ſame four plane angles, placed in the ſame order.

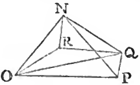
Take three plane angles A, B, C, of which A is not leſs than either of the other two, and ſuch, that A and B together are leſſer than two right angles; and by Problem 1. and its Corollary, find a fourth angle D ſuch, that any three whatever of the angles A, B, C, D be greater than the remaining angle, and ſuch, that A and B together be not leſſer than C and D together. and by Problem 2. find a fifth angle E ſuch, that any two of the angles A, B, E be greater than the third, and alſo that any two of the angles C, D, E be greater than the third. and becauſe A and B together are leſſer than two right angles, the double of A and B together ſhall be leſſer than four right angles. but A and B together are greater than the angle E, wherefor the double of A and B together ſhall be greater than the three angles A, B, E together, which three ſhall conſequently be leſſer than four right angles; and every two of the ſame angles A, B, E are greater than the third; therefor, by Prop. 23. 11. a ſolid angle can be



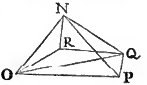


be made contained by three plane angles equal to the angles A, B, E, each to each. Let this be the angle F contained by the three plane angles GFH, HFK, GFK which are equal to the angles A, B, E, each to each. and becauſe the angles C, D together are not greater than the angles A, B together, the angles C, D, E ſhall not be greater than the angles A, B, E. but theſe laſt three are leſſer than four right angles, as has been demonſtrated, wherefor alſo the angles C, D, E are together leſs than four right angles, and every two of them are greater than the third; therefor a ſolid angle may be made which ſhall be contained by three plane angles equal to the angles C, D, E, each to each[a]. and by Prop. 26. 11. at the point F in the ſtraight a. 23. 11. line FG a ſolid angle can be made equal to that which is contained by the three plane angles that are equal to the angles C, D, E. let this be made, and let the angle GFK, which is equal to E, be one of the three; and let KFL, GFL be the other two which are equal to the angles C, D, each to each. thus, there ſhall be a ſolid angle conſtituted at the point F contained by the four plane angles GFH, HFK, KFL, GFL which are equal to the angles A, B, C, D, each to each.

Again, Find another angle M ſuch, that every two of the three angles A, B, M be greater than the third, and alſo every two of the three C, D, M be greater than the third. and, as in the preceeding part, it ſhall be demonſtrated that the three A, B, M are leſſer than four right angles, as alſo that the three C, D, M are leſſer than four right angles. Make therefor [a] a ſolid angle at N contained by the three plane angles ONP, PNQ, ONQ, which are equal to A, B, M, each to each. and by Prop. 26. 11. make



make at the point N in the ſtraight line ON a ſolid angle contained by three plane angles of which one is the angle ONQ equal to M, and the other two are the angles QNR, ONR which are equal to the angles C, D, each to each. thus at the point N there ſhall be a ſolid angle contained by the four plane angles ONP, PNQ, QNR, ONR which are equal to the angles A, B, C, D, each to each. and that the two ſolid angles at the points F, N, each of which is contained by the above named four plane angles, are not equal to one another, or that they cannot coincide, will be plain by conſidering that the angles GFK, ONQ; that is, the angles E, M are unequal by the conſtruction, and therefor the ſtraight lines GF, FK cannot coincide with ON, NQ, nor conſequently can the ſolid angles, which therefor are unequal.



And becauſe from the three given plane angles A, B, C there can be found innumerable other angles that will ſerve the ſame purpoſe with the angle D, and again from D or any one of theſe others, and the angles A, B, C, there may be found innumerable angles, ſuch as E or M; it is plain that innumerable other ſolid angles may be conſtituted which are each contained by the ſame four plane angles, and all of them unequal to one another. Q. E. D.

And from this it appears that Clavius and other Authors are miſtaken who aſſert that thoſe ſolid angles are equal which are contained by the ſame number of plane angles that are equal to one another, each to each. alſo it is plain that the 26. Prop. of Book 11. is by no means ſufficiently demonſtrated, becauſe the equality of two ſolid angles, whereof each is contained by three plane angles which are equal to one another, each to each, is only aſſumed, and not demonſtrated.

PROP. I.

PROP. I. B. XI.

The words at the end of this, "for a ſtraight line cannot meet a "ſtraight line in more than one point," are left out, as an addition by ſome unſkilful hand; for this is to be demonſtrated, not aſſumed.

PROP. II. B. XI.

This Propoſition ſeems to have been changed and vitiated by ſome Editor; for all the figures defined in the 1. Book of the Elements, and among them triangles, are, by the Hypotheſis, plane figures; that is, ſuch as are deſcribed in a plane; wherefor the ſecond part of the Enuntiation needs no Demonſtration. beſides a convex ſuperficies may be terminated by three ſtraight lines meeting one another. the thing that ſhould have been demonſtrated is, that two, or three ſtraight lines, that meet one another, are in one plane. and as this is not ſufficiently done, the Enuntiation and Demonſtration are changed into thoſe now put into the Text.

PROP. III. B. XI.

In this Propoſition the following words near to the end of it are left out, viz. "therefor DEB, DFB are not ſtraight lines, in the like "manner it ſhall be demonſtrated that there can be no other ſtraight "line between the points D, B." becauſe from this that two lines include a ſpace, it only follows that one of them is not a ſtraight line. and the force of the argument lies in this, viz. if the common ſection of the planes be not a ſtraight line, then two ſtraight lines could include a ſpace, which is abſurd; therefor the common ſection is a ſtraight line.

PROP. IV. B. XI.

The words "and the triangle AED to the triangle BEC" are omitted, becauſe the whole concluſion of the 4. Prop. B. 1. has been ſo often repeated in the preceeding Books, it was needleſs to repeat it here.

PROP. V. B. XI.

In this, near to the end, ἐπιπέδῳ, ought to be left out in the Greek text. and the word "plane" is rightly left out in the Oxford Edition of Commandine's Tranſlation.

PROP. VII. B. XI.

This Propoſition has been put into this Book by ſome unſkilful Editor, as is evident from this, that ſtraight lines which are drawn from one point to another in a plane, are, in the preceeding Books, ſuppoſed to be in that plane. and if they were not, ſome Demonſtrations in which one ſtraight line is ſuppoſed to meet another would not be concluſive, becauſe theſe lines would not meet one another. for inſtance, in Prop. 30. B. 1. the ſtraight line GK would not meet EF, if GK were not in the plane in which are the parallels AB, CD, and in which, by Hypotheſis, the ſtraight line EF is. beſides, this 7. Propoſition is demonſtrated by the preceeding 3. in which the very thing which is propoſed to be demonſtrated in the 7. is twice aſſumed, viz. that the ſtraight line drawn from one point to another in a plane, is in that plane; and the ſame thing is aſſumed in the preceeding 6. Prop. in which the ſtraight line which joins the points B, D that are in the plane to which AB and CD are at right angles, is ſuppoſed to be in that plane. and the 7. of which another Demonſtration is given, is kept in the Book merely to preſerve the number of the Propoſitions; for it is evident from the 7. and 35. Definitions of the 1. Book, tho' it had not been in the Elements.

PROP. VIII. B. XI.

In the Greek, and in Commandine's and Dr. Gregory's Tranſlations, near to the end of this Propoſition, are the following words, " but DC is in the plane thro' BA, AD" inſtead of which in the Oxford edition of Commandine's tranſlation is rightly put " but DC is " in the plane thro' BD, DA." but all the Editions have the following words, viz. " becauſe AB, BD are in the plane thro' BD, DA; and DC ~~" is in the plane in which are AB, BD,"~~ which are manifeſtly corrupted, or have been added to the Text; for there was not the leaſt neceſſity to go ſo far about to ſhew that DC is in the ſame plane in which are BD, DA, becauſe it immediately follows from Prop. 7. preceeding, that BD, DA are in the plane in which are the parallels AB, CD. therefor inſtead of theſe words there ought only to be " becauſe all three are in the plane in which are the parallels AB, CD."

PROP. XV. B. XI.

After the words, " and becauſe BA is parallel to AH," the following are added " for each of them is parallel to DE, and are not " both in the ſame plane with it," as being manifeſtly forgotten to be put into the Text.

PROP. XVI. B. XI.

In this, near to the end, inſtead of the words " but ſtraight lines " which meet neither way" ought to be read " but ſtraight lines in " the ſame plane which produced meet neither way." becauſe tho' in citing this Definition in Prop. 27. B. 1. it was not neceſſary to mention the words, "in the ſame plane" all the ſtraight lines in the Books preceeding this being in the ſame plane; yet here it was quite neceſſary.

PROP. XX. B. XI.

In this, near the beginning, are the words, "but if not, let BAC "be the greater." but the angle BAC may happen to be equal to one of the other two. wherefor this place ſhould be read thus, "but if "not, let the angle BAC be not leſſer than either of the other two, "but greater than DAB."

At the end of this Propoſition it is ſaid, "in the ſame manner it "ſhall be demonſtrated," tho' there is no need of any Demonſtration; becauſe the angle BAC being not leſſer than either of the other two, it is evident that BAC together with one of them is greater than the other.

PROP. XXII. B. XI.

And likewiſe in this, near the beginning, it is ſaid, "but if not, let "the angles at B, E, H be unequal, and let the angle at B be greater "than either of thoſe at E, H." which words manifeſtly ſhew this place to be vitiated, becauſe the angle at B may be equal to one of the other two. they ought therefor to be read thus, "but if not, let the "angles at B, E, H be unequal, and let the angle at B be not leſſer "than either of the other two at E, H. therefor the ſtraight line AC "is not leſſer than either of the two DF, GK."

PROP. XXIII. B. XI.

The Demonſtration of this is made ſomething ſhorter, by not repeating in the third Caſe the things which were demonſtrated in the firſt; and by making uſe of the conſtruction which Campanus has given; but he does not demonſtrate the ſecond and third Caſes. the Conſtruction and Demonſtration of the third Caſe are made a little more ſimple than in the Greek text.

PROP. XXIV. B. XI.

The word "ſimilar" is added to the Enuntiation of this Propoſition, becauſe the planes which contain the ſolids which are to be demonſtrated to be equal to one another, in the 25. Propoſition, ought to be ſimilar and equal; that the equality of the ſolids may be inferred from Prop. C. of this Book. and in the Oxford Edition of Commandine's Tranſlation a Corollary is added to Prop. 24. to ſhew that the parallelograms mentioned in this Propoſition are ſimilar, that the equality of the ſolids in Prop. 25. may be deduced from the 10. Def. of B. 11.

PROP. XXV. and XXVI. B. XI.

In the 25. Prop. ſolid figures which are contained by the ſame number of ſimilar and equal plane figures, are ſuppoſed to be equal to one another. and it ſeems that Theon, or ſome other Editor, that he might ſave himſelf the trouble of demonſtrating the ſolid figures mentioned in this Propoſition to be equal to one another, has inſerted the 10. Def. of this Book, to ſerve inſtead of a Demonſtration; which was very ignorantly done.

Likewiſe in the 26. Prop. two ſolid angles are ſuppoſed to be equal, if each of them be contained by three plane angles which are equal to one another, each to each. and it is ſtrange enough, that none of the Commentators on Euclid, have, as far as I know, perceived that ſomething is wanting in the Demonſtrations of theſe two Propoſitions. Clavius, indeed, in a Note upon the 11. Def. of this Book, affirms,

that

that it is evident that theſe ſolid angles are equal which are contained by the ſame number of plane angles, equal to one another, each to each, becauſe they will coincide, if they be conceived to be placed within one another; but this is ſaid without any proof, nor is it always true, except when the ſolid angles are contained by three plane angles only, which are equal to one another, each to each. and in this caſe the Propoſition is the ſame with this, that two ſpherical triangles that are equilateral to one another, are alſo equiangular to one another, and can coincide; which ought not to be granted without a Demonſtration. Euclid does not aſſume this in the caſe of rectilineal triangles, but demonſtrates in Prop. 8. B. 1. that triangles which are equilateral to one another are alſo equiangular to one another; and from this their total equality appears by Prop. 4. B. 1. and Menelaus, in the 4. Prop. of his 1. Book of Sphaerics, explicitly demonſtrates that ſphaerical triangles which are mutually equilateral, are alſo equiangular to one another; from which it is eaſy to ſhew that they ſhall coincide, providing they have their ſides diſpoſed in the ſame order and ſituation.

To ſupply theſe defects, it was neceſſary to add the three Propoſitions marked A, B, C to this Book. for the 25. 26. and 28. Propoſitions of it, and conſequently eight others, viz. the 27. 31. 32. 33. 34. 36. 37. and 40. of the ſame, which depend upon them, have hitherto ſtood upon an infirm foundation; as alſo, the 8. 12. Cor. of 17. and 18. of the 12. Book, which depend upon the 9. Definition. for it has been ſhewn in the Notes on Def. 10. of this Book, that ſolid figures which are contained by the ſame number of ſimilar and equal plane figures, as alſo ſolid angles that are contained by the ſame number of equal plane angles are not always equal to one another.

It is to be obſerved that Tacquet, in his Euclid, defines equal ſolid angles to be ſuch, " as being put within one another ſhall coincide." but this is an Axiom, not a Definition, for it is true of all magnitudes whatever. he made this uſeleſs Definition, that by it he might demonſtrate the 36. Prop. of this Book without the help of the 35. of the ſame. concerning which Demonſtration, ſee the Note upon Prop. 36.

PROP. XXVIII. B. XI.

In this it ought to have been demonſtrated, not aſſumed, that the Diagonals are in one plane. Clavius has ſupplied this defect.

PROP. XXIX. B. XI.

There are three Caſes of this Propoſition; the firſt is when the two parallelograms oppoſite to the baſe AB have a ſide common to both; the ſecond is when theſe parallelograms are ſeparated from one another; and the third, when there is a part of them common to both; and to this laſt only, the Demonſtration that has hitherto been in the Elements does agree. The firſt Caſe is immediately deduced from the preceeding 28. Propoſition, which ſeems for this purpoſe to have been premiſed to this 29. for it is neceſſary to none but to it, and to the 40. of this Book, as we now have it, to which laſt it would, without doubt, have been premiſed, if Euclid had not made uſe of it in the 29. but ſome unſkilful Editor has taken it away from the Elements, and has mutilated Euclid's Demonſtration of the other two Caſes, which is now reſtored, and ſerves for both at once.

PROP. XXX. B. XI.

In the Demonſtration of this, the oppoſite planes of the ſolid CP, in the figure in this Edition; that is, of the ſolid CO in Commandine's figure, are not proved to be parallel; which is proper to do for the ſake of learners.

PROP. XXXI. B. XI.

There are two Caſes of this Propoſition; the firſt is when the inſiſting ſtraight lines are at right angles to the baſes; the other when they are not. the firſt Caſe is divided again into two others, one of which is when the baſes are equiangular parallelograms; the other when they are not equiangular. the Greek Editor makes no mention of the firſt of theſe two laſt Caſes, but has inſerted the Demonſtration of it as a part of that of the other. and therefor ſhould have taken notice of it in a Corollary; but we thought it better to give theſe two Caſes ſeparately. the Demonſtration alſo is made ſomething ſhorter by following the way Euclid has made uſe of in Prop. 14. B. 6. beſides, in the Demonſtration of the caſe in which the inſiſting ſtraight lines are not at right angles to the baſes, the Editor does not prove that the ſolids deſcribed in the conſtruction are parallelepipeds, which it is not to be thought that Euclid neglected. alſo the words, "whoſe inſiſting "ſtraight lines are not in the ſame ſtraight lines," have been added by ſome unſkilful hand; for they may be in the ſame ſtraight lines.

PROP. XXXII. B. XI.

The Editor has forgot to order the parallelogram FH to be applied in the angle FGH equal to the angle LCG, which is neceſſary. Clavius has ſupplied this.

Alſo, in the conſtruction, it is required to compleat the ſolid whoſe baſe is FH, and its altitude the ſame with that of the ſolid CD; but this does not determine the ſolid to be compleated, ſince there may be innumerable ſolids upon the ſame baſe, and of the ſame altitude. it ought therefor to be ſaid "compleat the ſolid whoſe baſe is FH, and one of its inſiſting ſtraight lines is FD. the ſame correction muſt be made in the following Propoſition 34.

PROP. D. B. XI.

It is very probable that Euclid gave this Propoſition a place in the Elements, ſince he gave the like Propoſition concerning equiangular parallelograms in the 23. B. 6.

PROP. XXXIV. B. XI.

In this the words, *ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν*, "whoſe inſiſting ſtraight lines are not in the ſame ſtraight lines" are thrice repeated; but theſe words ought either to be left out, as they are by Clavius, or in place of them ought to be put "whether the inſiſting ſtraight lines be, or be not, in the ſame ſtraight lines." for the other Caſe is without any reaſon excluded. alſo the words, *ὧν τὰ ὕψη*, "whoſe altitudes" are twice put for *ὧν αἱ ἐφεστῶσαι*, "whoſe inſiſting ſtraight lines;" which is a plain miſtake. for the altitude is always at right angles to the baſe.

PROP. XXXV. B. XI.

The angles ABH, DEM are demonſtrated to be right angles in a ſhorter way than in the Greek; and in the ſame way ACH, DFM may be demonſtrated to be right angles. alſo the repetition of the ſame Demonſtration, which begins with "in the ſame manner", is left out, as it was probably added to the Text by ſome Editor; for the words, "we ſhall demonſtrate in the ſame manner" are not inſerted except when the Demonſtration is not given, or when it is ſomething different from the other, if it be given, as in Prop. 26. of this Book. Campanus has not this repetition.

We have given another Demonſtration of the Corollary, beſides the one in the Original, by help of which the following 36. Propoſition may be demonſtrated without the 35.

PROP. XXXVI. B. XI.

Tacquet in his Euclid demonſtrates this Propoſition without the help of the 35. but it is plain that the ſolids mentioned in the Greek Text in the Enuntiation of the Propoſition as equiangular, are ſuch whoſe ſolid angles are contained by three plane angles equal to one another, each to each; as is evident from the conſtruction. Now Tacquet does not demonſtrate, but aſſumes theſe ſolid angles to be equal to one another; for he ſuppoſes the ſolids to be already made, and does not give the conſtruction by which they are made. but, by the ſecond Demonſtration of the preceeding Corollary, his Demonſtration is rendered legitimate likewiſe in the Caſe where the ſolids are conſtructed as in the Text.

PROP. XXXVII. B. XI.

In this it is aſſumed that the ratios which are triplicate of thoſe ratios which are the ſame to one another, are likewiſe the ſame to one another; and that thoſe ratios are the ſame to one another, whoſe triplicate ratios are the ſame to one another; but this ought not to be granted without a Demonſtration, nor did Euclid aſſume the firſt and eaſieſt of theſe two Propoſitions, but demonſtrated it in the caſe of duplicate ratios, in the 22. Prop. B. 6. on this account another Demonſtration is given of this Propoſition like to that which Euclid gives in Prop. 22. B. 6. as Clavius has done.

PROP. XXXVIII. B. XI.

When it is required to draw a perpendicular from a point in one plane which is at right angles to another plane, unto this laſt plane, it is done by drawing a perpendicular from the point to the common ſection of the planes; for this perpendicular ſhall be perpendicular to the plane, by Def. 4. of this Book. and it would be fooliſh in this caſe to do it by the 11. Propoſition of the ſame. but Euclid[*], Apollonius, and other Geometers, when they have occaſion for this Problem, bid draw a perpendicular from the point to the plane, and conclude that it will fall upon the common ſection of the planes, becauſe this is the ~~very ſame thing as if they had made uſe of the conſtruction above men~~tioned, and then concluded that the ſtraight line ſhall be perpendicular to the plane; but is expreſſed in fewer words. ſome Editor who has not perceived this, thought it was neceſſary to add this Propoſition to the 11. Book, which never can be of any uſe. and its being near to the end of the Book among Propoſitions with which it has no connexion, is a mark of its having been added to the Text.

[*] 17. 12.

PROP. XXXIX. B. XI.

In this it is ſuppoſed that the ſtraight lines which biſſect the ſides of the oppoſite planes, are in one plane, which ought to have been demonſtrated; as is now done.

B. XII.

THE learned Mr. Moor, Profeſſor of Greek in the Univerſity of Glaſgow, obſerved to me that it plainly appears from Archimedes Epiſtle to Doſitheus prefixed to his Books of the Sphaere and Cylinder, which Epiſtle he has reſtored from antient Manuſcripts, that Eudoxus was the Author of the chief Propoſitions in this twelfth Book.

PROP. II. B. XII.

At the beginning of this it is ſaid, " if it be not ſo, the ſquare of " BD ſhall be to the ſquare of FH, as the circle ABCD is to ſome " ſpace either leſſer than the circle EFGH, or greater than it." and the like is to be found near to the end of this Propoſition, as alſo in Prop. 5. 11. 12. 18. of this Book. concerning which it is to be obſerved, that in the Demonſtration of Theorems, it is ſufficient, in this and the like caſes, that a thing made uſe of in the reaſoning can poſſibly exiſt, providing this be evident, tho it cannot be exhibited or found by a Geometrical conſtruction. ſo in this place it is aſſumed that there can be a fourth proportional to theſe three magnitudes, viz. the ſquares of BD, FH, and the circle ABCD; becauſe it is evident that there is ſome ſquare equal to the circle ABCD, tho' it cannot be found geometrically; and to the three rectilineal figures, viz. the ſquares of BD, FH, and the ſquare which is equal to the circle ABCD, there is a fourth ſquare proportional; becauſe to the three ſtraight lines which are their ſides there is a fourth ſtraight line proportional[a]. and this fourth ſquare, or a ſpace equal to it, is the ſpace which in this Propoſition is denoted by the letter S. and the like is to be underſtood in the other places above cited. and it is probable this has been ſhewn by Euclid, but left out by ſome Editor; for the Lemma which ſome unſkilful hand has added to this Propoſition explains nothing of it.

[a] 12. 6.

PROP. III. B. XII.

In the Greek Text and the Tranſlations, it is ſaid, " and becauſe " the two ſtraight lines BA, AC which meet one another" &c. here the angles BAC, KHL are demonſtrated to be equal to one another by 10. Prop. B. 11. which had been done before. becauſe the triangle EAG was proved to be ſimilar to the triangle KHL. this repetition is left out, and the triangles BAC, KHL are proved to be ſimilar in a ſhorter way by Prop. 21. B. 6.

PROP. IV. B. XII.

A few things in this are more fully explained than in the Greek Text.

PROP. V. B. XII.

In this, near to the end, are the words ὡς ἔμπροσθεν ἐδείχθη, "as "was before ſhewn," and the ſame are found again in the end of Prop. 18. of this Book; but the Demonſtration referred to, except it be the uſeleſs Lemma annexed to the 2. Prop. is no where in theſe Elements, and has been perhaps left out by ſome Editor who has forgot to cancel theſe words alſo.

PROP. VI. B. XII.

A ſhorter Demonſtration is given of this; and that which is in the Greek Text may be made ſhorter by a ſtep than it is. for the Author of it makes uſe of the 22. Prop. of B. 5. twice, whereas once would have ſerved his purpoſe; becauſe that Propoſition extends to any number of magnitudes which are proportionals taken two and two, as well as to three which are proportional to other three.

COR. PROP. VIII. B. XII.

The Demonſtration of this is imperfect, becauſe the pyramids into which thoſe upon multangular baſes are divided, ought to be proved to be ſimilar to one another, each to each; as is done in the like caſe in the 12. Prop. following. from this it appears that this Corollary has been added to the Text. and theſe partial pyramids being proved ſimilar, it follows they have, each to each, the triplicate ratio of their homologous ſides, to wit, of thoſe which are alſo the ſides of the polygons which are the baſes of the whole pyramids. and the polygons being ſimilar, their ſides have, each to each, the ſame ratio. wherefor alſo their triplicate ratios, that is, the ratios of the partial pyramids, are the ſame to one another. and by the 12. Prop. of B. 5. the ratio of the total pyramids is the ſame to that of the partial; that is, to the triplicate ratio of their homologous ſides.

PROP. XI. and XII. B. XII.

The order of the letters of the Alphabet is not obſerved in theſe two Propoſitions, according to Euclid's manner, and is now reſtored. by which means the firſt part of Prop. 12. may be demonſtrated in the ſame words with the firſt part of Prop. 11. on this account the Demonſtration of that firſt part is left out, and aſſumed from Prop. 11.

PROP. XIII. B. XII.

In this Propoſition the common ſection of a plane parallel to the baſes of a cylinder, with the cylinder itſelf is ſuppoſed to be a circle, and it was thought proper briefly to demonſtrate it; from whence it is ſufficiently manifeſt that this plane divides the cylinder into two others. and the ſame thing is to be underſtood to be ſupplied in Prop. 14.

PROP. XV. B. XII.

" And compleat the cylinders AX, EO." both the Enuntiation and Expoſition of the Propoſition repreſent the cylinders as well as the cones as already deſcribed. wherefor the reading ought rather to be " and let the cones be ALC, ENG; and the cylinders AX, EO."

PROP. XVII. B. XII.

In the Enuntiation of this Propoſition the Greek words, ἐις τὴν μείζονα σφαῖραν ϛερεὸν, πολύεδρον ἐγγράψαι, μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν, are thus tranſlated by Commandine and others, " in majori ſolidum polyedrum deſcribere quod minoris " ſphaerae ſuperficiem non tangat;" that is, " to deſcribe in the greater " ſphere a ſolid polyhedron which ſhall not meet the ſuperficies of the " leſſer ſphere." whereby they refer the words κατὰ τὴν ἐπιφάνειαν to theſe next to them τῆς ἐλάσσονος σφαίρας. but they ought by no means to be thus tranſlated, for the ſolid polyhedron doth not only meet the ſuperficies of the leſſer ſphere, but pervades the whole of that ſphere. therefor the foreſaid words are to be referred to τὸ ϛερεὸν πολύεδρον, and ought thus to be tranſlated, viz. to deſcribe in the greater ſphere a ſolid polyhedron whoſe ſuperficies ſhall not meet the leſſer ſphere; as the meaning of the Propoſition neceſſarily requires.

The Demonſtration of the Propoſition is ſpoiled and mutilated. for ſome eaſy things are very explicitly demonſtrated, while others not ſo obvious are not ſufficiently explained; for example, when it is affirmed that the ſquare of KB is greater than the double of the ſquare of BZ, in the firſt Demonſtration; and that the angle BZK is obtuſe, in the ſecond. both which ought to have been demonſtrated. beſides, in the firſt Demonſtration it is ſaid " draw KΩ from the point K per- " pendicular to BV;" whereas it ought to have been ſaid, " join KV," and it ſhould have been demonſtrated that KV is perpendicular to BV. for it is evident from the figure in Hervagius's and Gregory's Editions, and from the words of the Demonſtration, that the Greek Editor did not perceive that the perpendicular drawn from the point K to the ſtraight line BV muſt neceſſarily fall upon the point V, for in the figure it is made to fall upon the point Ω a different point from V, which is likewiſe ſuppoſed in the Demonſtration. Commandine ſeems to have been aware of this; for in his figure he marks one and the ſame point with the two letters V, Ω; and before Commandine, the learned John Dee in the Commentary he annexes to this Propoſition in Henry Billingſley's Tranſlation of the Elements printed at London Ann. 1570, expreſly takes notice of this error, and gives a Demonſtration ſuited to the Conſtruction in the Greek Text, by which he ſhews that the perpendicular drawn from the point K to BV, muſt neceſſarily fall upon the point V.

Likewiſe it is not demonſtrated that the quadrilateral figures SOPT, TPRY, and the triangle YRX ſhall not meet the leſſer ſphere, as was neceſſary to have done. only Clavius, as far as I know, has obſerved this, and demonſtrated it by a Lemma, which is now premiſed to this Propoſition, ſomething altered and more briefly demonſtrated.

In the Corollary of this Propoſition it is ſuppoſed that a ſolid polyhedron is deſcribed in the other ſphere ſimilar to that which is deſcribed in the ſphere BCDE. but as the Conſtruction by which this may be done is not given, it was thought proper to give it, and to demonſtrate that the pyramids in it are ſimilar to thoſe of the ſame order in the ſolid polyhedron deſcribed in the ſphere BCDE.

From the preceeding Notes it is ſufficiently evident how much the Elements of Euclid, who was a moſt accurate Geometer, have been vitiated and mutilated by ignorant Editors. The opinion which the moſt part of learned men have entertained concerning the preſent Greek Edition, viz. that it is very little or nothing different from the genuine work of Euclid, has, without doubt deceived them, and made them leſs attentive and accurate in examining this Edition we now have, whereby it has happened that ſeveral errors, tho' ſome of them groſs enough, have eſcaped their notice from the age in which Theon lived to this preſent time. Upon this account there is ſome ground to hope that the pains we have taken in correcting theſe errors, and freeing the Elements as far as we could from blemiſhes, will not be unacceptable to good Judges who can diſcern when Demonſtrations are legitimate, and when they are not.

FINIS.

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